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A NEW TRIGONOMETRY

FOR SCHOOLS

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A NEW TRIGONOMETRY FOR SCHOOLS

PART II.

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PREFACE.

THE recent changes in the methods of teaching Elementary Mathematics (so largely due to the genius of Prof. Perry) have considerably affected Plane Trigonometry. Students are expected to have a good *practical* knowledge of the subject, while great skill in the solution of artificial problems and identities has ceased to be regarded as the aim and object of the subject.

This book has been written with a view to these changes and to supply the need felt for a School Trigonometry based on the use of *Four Figure Logarithms*, in which Logarithms, the Solution of Triangles and the more practical parts of the subject are introduced as early as possible. For this reason the expansions of $\sin(A+B)$, etc. and harder identities are deferred until after the Solution of Triangles, Heights and Distances, etc.

Seeing that incommensurable quantities are now omitted in Elementary Geometry and consequently no difficulty is found with the various theorems relating to arcs and sectors of circles, it has been thought advisable to place the Circular Measurement of Angles immediately after the measurement in degrees, etc.

Graphical Methods and *Squared Paper* are largely employed in the approximation to trigonometrical ratios of a given angle, in finding angles from given ratios, in the variations of trigonometrical expressions and logarithms.

Students are advised always to *check* their results in the Solution of Triangles, Heights and Distances, etc., by drawing figures to scale.

The more *theoretical* parts are treated with fulness for the benefit of those intending to proceed to higher branches of mathematics.

Part I includes Solution of Triangles, Heights and Distances, and Functions of Compound Angles, and is sufficient for the Oxford and Cambridge Junior Local, Mathematics I of the Woolwich and Sandhurst Examination, etc. It contains over 1200 examples.

Part II contains chapters on De Moivre's Theorem, the Exponential Theorem and the expansion of $\sin\theta$ and $\cos\theta$ in terms of θ, etc.

Considerable care has been given to the selection of examples, many of which are taken from recent Army and Navy Entrance and the various Cambridge Examinations.

An appendix on the *Slide Rule* will be found useful for students preparing for the Entrance Examinations to Woolwich and Sandhurst.

It is hoped that the sets of *Test Papers*, which have been very carefully graduated to fit in with the sequence of chapters in the book, will prove useful for purposes of revision. Harder questions will be found in the Miscellaneous Examples.

The examples have all been verified from the proof sheets and it is hoped that very few errors remain; in the use of four figure tables, answers vary slightly according to the precise method of working; *e.g.* log 4 is not exactly the same as 2 log 2; such variations occur chiefly in solving triangles when there are several formulae applicable; the authors have in many cases indicated which formulae should be used to obtain the answers in the book.

The authors wish to express their gratitude for many suggestions received from Mr T. Hyett of Cheltenham College.

CHELTENHAM COLLEGE,
September 1904.

CONTENTS.

PART II.

CHAPTER XIV.

PROPERTIES OF TRIANGLES (*continued*).

THE principal formulae in this chapter are the following:

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

$$= \frac{abc}{4\Delta},$$

$$r = \frac{\Delta}{s}$$

$$= (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

$$= \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

$$r_1 = \frac{\Delta}{s-a}$$

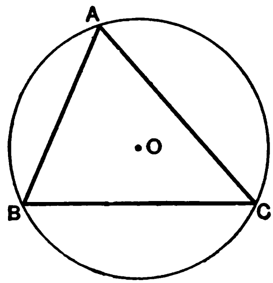
$$= s\tan\frac{A}{2} = (s-c)\cot\frac{B}{2} = (s-b)\cot\frac{C}{2}$$

$$= \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}} = \frac{c\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}$$

$$= 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Similar expressions may be found for r_2 and r_3.

142. *To find the radius of the circumcircle of a triangle.*



We have already proved in Art. 73,

that $$R = \frac{a}{2 \sin A}.$$

Now $$\frac{a}{2 \sin A} = \frac{abc}{4 \times \frac{1}{2} bc \sin A}$$
$$= \frac{abc}{4\Delta};$$
$$\therefore\ R = \frac{abc}{4\Delta}.$$

143. *To find the radius of the in-circle of a triangle.*

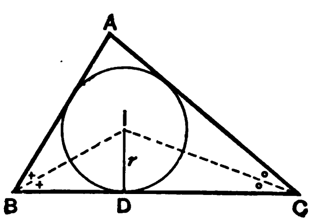
The in-centre I being found by bisecting two angles of the triangle by the lines BI, CI, a perpendicular ID is drawn to the side BC.

We have already proved in Arts. 79 and 80 that

$$r = \frac{\Delta}{s} = (s - a) \tan \frac{A}{2}.$$

Also

$$\frac{r}{a} = \frac{r}{IB} \cdot \frac{IB}{a} = \sin \frac{B}{2} \cdot \frac{\sin \frac{C}{2}}{\sin BIC}$$
$$= \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\sin\left(\pi - \frac{B+C}{2}\right)}$$
$$= \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}},$$



$$\therefore r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = \frac{2R \sin A \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$$

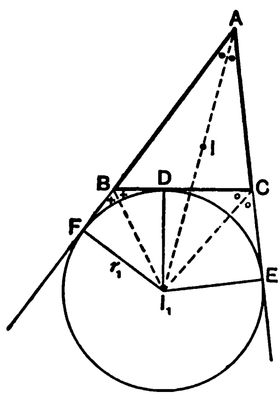
$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

Note that $$\frac{IB}{a} = \frac{\sin \frac{C}{2}}{\cos \frac{A}{2}},$$

$$\therefore IB = 4R \sin \frac{A}{2} \sin \frac{C}{2}, \text{ etc.}$$

144. *To find the radius of an escribed circle of a triangle.*

The *e*-centre opposite the angle A is found by bisecting the exterior angles CBF, BCE by the lines BI_1 and CI_1. Perpendiculars I_1D, I_1E, I_1F are then drawn to the sides of the triangle.



$$\Delta = ABI_1C - \text{area of } BI_1C$$
$$= ABI_1 + ACI_1 - BI_1C$$
$$= \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1$$
$$= r_1\left(\frac{b+c-a}{2}\right)$$
$$= r_1\left(\frac{b+c+a}{2} - a\right)$$
$$= r_1(s-a);$$

$$\therefore \mathbf{r_1} = \frac{\Delta}{\mathbf{s-a}}.$$

Similarly if r_2 and r_3 are the radii of the *e*-circles opposite the angles B and C respectively

$$r_2 = \frac{\Delta}{s-b}, \qquad r_3 = \frac{\Delta}{s-c}.$$

145. Let $BD(=BF)=x$; $CD(=CE)=a-x$.

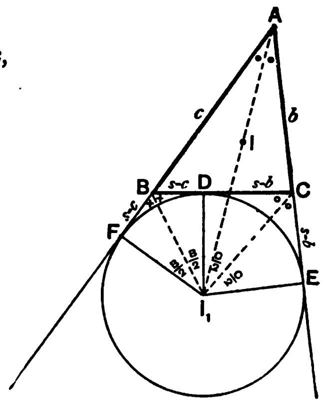
$\therefore AF=c+x$ and $AE=b+a-x$,

$$\therefore c+x=b+a-x,$$

$$\therefore BD(=BF)=x=\frac{b+a-c}{2}=s-c,$$

$$\therefore AF(=AE)=c+x=s,$$

$$CD(=CE)=a-x=s-b.$$



Since the angles BFI_1 and BDI_1 are right angles, it follows that the points I_1FBD are concyclic.

$$\therefore A\hat{B}C(=B)=F\hat{I_1}D$$

$$\therefore F\hat{I_1}B=\frac{B}{2}.$$

Similarly $$E\hat{I_1}C=\frac{C}{2}.$$

If I is the in-centre, then AII_1 is a straight line.

$$r_1=AF\tan\frac{A}{2}$$

$$=s\tan\frac{A}{2}.$$

Also $$r_1=BF\cot FI_1B=(s-c)\cot\frac{B}{2}$$

$$=EC\cot EI_1C=(s-b)\cot\frac{C}{2}.$$

Similarly $$r_2=s\tan\frac{B}{2}=(s-a)\cot\frac{C}{2}=(s-c)\cot\frac{A}{2}$$

$$r_3=s\tan\frac{C}{2}=(s-a)\cot\frac{B}{2}=(s-b)\cot\frac{A}{2}.$$

146. $$\frac{r_1}{a}=\frac{r_1}{I_1B}\cdot\frac{I_1B}{a}=\sin FBI_1\cdot\frac{\sin BCI_1}{\sin BI_1C}$$

$$=\cos\frac{B}{2}\cdot\frac{\cos\frac{C}{2}}{\sin\frac{B+C}{2}};$$

$$\therefore\ r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=\frac{b\sin A}{\sin B}\cdot\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}\text{ or }\frac{c\sin A}{\sin C}\cdot\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=\frac{b\sin\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}}\text{ or }\frac{c\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}.$$

Also $$r_1=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=\frac{2R\sin A\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

with corresponding expressions for r_2 and r_3.

Note that $$\frac{I_1B}{a}=\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

$$\therefore\ I_1B=4R\sin\frac{A}{2}\cos\frac{C}{2},\text{ etc.}$$

147. Ex. 1. Prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

$$\begin{aligned}\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}&=\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\\&=\frac{3s-(a+b+c)}{\Delta}\\&=\frac{s}{\Delta}\\&=\frac{1}{r}.\end{aligned}$$

Ex. 2. Show that $\frac{b^2-c^2}{2}\cdot\frac{\sin B\sin C}{\sin(B-C)}=\Delta$.

$$\begin{aligned}\text{Expression}&=\frac{4R^2(\sin^2 B-\sin^2 C)}{2}\cdot\frac{\sin B\sin C}{\sin(B-C)}\\&=\frac{4R^2\sin(B+C)}{2}\cdot\sin B\sin C\\&=\frac{4R^2\sin A\sin B\sin C}{2}\\&=\tfrac{1}{2}bc\sin C\\&=\Delta.\end{aligned}$$

EXAMPLES XXXV.

Prove that

1. $\Delta=\sqrt{rr_1r_2r_3}$.
2. $\Delta=\dfrac{arr_1}{r_1-r}$.
3. $\Delta=\dfrac{(b+c)rr_1}{r+r_1}$.
4. $\Delta=\dfrac{ar_2r_3}{r_2+r_3}$.
5. $\Delta=\dfrac{rr_1(r_2-r_3)}{b-c}$.
6. $\Delta=\dfrac{rr_2\sqrt{r_1+r_3}}{\sqrt{r_2-r}}$.
7. $\Delta=r_2r_3\tan\dfrac{A}{2}$.
8. $\Delta=rr_1\cot\dfrac{A}{2}$.

9. $\Delta = r_1 r_2 r_3 / \sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}$.

10. $\Delta = r \sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}$.

11. $\Delta = \dfrac{r}{2R^{\frac{1}{2}}} \sqrt{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}$.

12. $rs^2 = r_1 r_2 r_3$.

13. $r_1 r_2 r_3 = r^3 \cot^2 \dfrac{A}{2} \cot^2 \dfrac{B}{2} \cot^2 \dfrac{C}{2}$.

14. $4Rrs = abc$.

15. $4R = r_1 + r_2 + r_3 - r$.

16. $r_3 = r \cot \dfrac{A}{2} \cot \dfrac{B}{2}$.

17. $rr_1 = r_2 r_3 \tan^2 \dfrac{A}{2}$.

18. $a(rr_1 + r_2 r_3) = b(rr_2 + r_3 r_1) = c(rr_3 + r_1 r_2)$.

19. $\left(\dfrac{r_1}{r} - 1\right)\left(\dfrac{r_2}{r} - 1\right)\left(\dfrac{r_3}{r} - 1\right) = \dfrac{4R}{r}$.

20. $2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)$.

21. $2(R + r) = a \cot A + b \cot B + c \cot C$.

22. $\Delta^2 \left(\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2}\right) = a^2 + b^2 + c^2$.

23. $4Rr + r^2 = ab + bc + ca - s^2$.

24. $\dfrac{1}{c \sin B} + \dfrac{1}{a \sin C} + \dfrac{1}{b \sin A} = \dfrac{1}{r}$.

25. $4\Delta(\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$.

26. $(b - c) r_2 r_3 + (c - a) r_3 r_1 + (a - b) r_1 r_2 = 0$.

27. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a \sin A + b \sin B + c \sin C}{4\Delta}$.

28. $2R(1 - \cos A) = r_1 - r$.

29. $\dfrac{\cos A}{c \sin B} + \dfrac{\cos B}{a \sin C} + \dfrac{\cos C}{b \sin A} = \dfrac{1}{R}$.

30. Find the radius of the inscribed circle of a triangle whose sides are 706, 690 and 240 feet.

31. If the sides of a triangle are 3, 4, 5 inches in length, in what ratio do the points of contact of the inscribed circle divide the sides?

32. If the sides of a triangle are 5, 6 and 9 centimetres in length, find the radius of the circum-circle.

Prove that

33. $r_1(\cos B - \cos C) + r_2(\cos C - \cos A) + r_3(\cos A - \cos B) = 0.$

34. $\dfrac{r_3^2}{4R - r_1 - r_2} = r_3 + \dfrac{r_1 r_2}{r_1 + r_2}.$

35. $p_1 \cos A + p_2 \cos B + p_3 \cos C = 2R(1 + \cos A \cos B \cos C),$

where p_1, p_2, p_3 are the perpendiculars from A, B, C on the opposite sides of the triangle ABC.

36. $abc + (a-b)(b-c)(c-a) = 4Rr(a \cos C + b \cos A + c \cos B).$

37. $8R^2(1 + \cos A \cos B \cos C) = a^2 + b^2 + c^2.$

38. $a^3r^2 - 2a^2\Delta r + a(r^4 + 4r^3R + \Delta^2) - 4\Delta R r^2 = 0.$

39. If p is the perpendicular from the angle A on to BC,

$$p = r \operatorname{cosec} \frac{A}{2} \sqrt{(1 + \cos B)(1 + \cos C)}.$$

40. If in the ambiguous case of the solution of a triangle where a, c and C are given, the two values of b are b_1 and b_2 and r_1, r_2 be the radii of the corresponding in-circles, prove that

$$\left(\frac{b_1}{r_1} - \cot \frac{C}{2}\right)\left(\frac{b_2}{r_2} - \cot \frac{C}{2}\right) = 1,$$

and $$r_1 r_2 = a(a-c)\sin^2\frac{C}{2}.$$

41. $$\Delta = r_1 r_2 \sqrt{\frac{4R - r_1 - r_2}{r_1 + r_2}}.$$

42. Prove that if θ is the angle at which the perpendicular from the vertex A to the side BC of a triangle ABC cuts the inscribed circle, then

$$\cos\theta = \sin\tfrac{1}{2}(B - C) \operatorname{cosec} \tfrac{1}{2} A.$$

43. Prove that

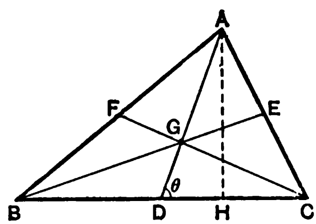
$$\sin^2 A + \sin B \sin C \cos A = 2\Delta^2 (a^2 + b^2 + c^2)/a^2 b^2 c^2.$$

44. Prove that if the bisector of the angle C cuts AB in D and the circum-circle in E,

$$CE/DE = (a+b)^2/c^2.$$

148. Medians.

If AD, BE and CF are the medians, then G the point of intersection is known as the *Centroid*, and by Elementary Geometry $\frac{AG}{GD} = \frac{2}{1}$, etc.



Also

$$2AD^2 + 2BD^2 = AB^2 + AC^2,$$

$$\therefore\ 2AD^2 = c^2 + b^2 - \frac{a^2}{2}$$

or

$$AD^2 = \frac{1}{2}\left(b^2 + c^2 - \frac{a^2}{2}\right).$$

Similarly

$$BE^2 = \frac{1}{2}\left(c^2 + a^2 - \frac{b^2}{2}\right)$$

$$CF^2 = \frac{1}{2}\left(a^2 + b^2 - \frac{c^2}{2}\right).$$

149. If θ is the angle that AD makes with BC, and AH is perpendicular to BC,

$$DH = \tfrac{1}{2}[(BD + DH) - (DC - DH)]$$
$$= \tfrac{1}{2}(BH - HC);$$

$$\therefore\ \cot\theta = \frac{DH}{AH} = \frac{1}{2}\frac{BH - HC}{AH}$$
$$= \tfrac{1}{2}(\cot B - \cot C).$$

Also since

$$\cos\theta = \frac{CD^2 + DA^2 - AC^2}{2DC\,.\,AD} = \frac{\frac{a^2}{4} + \left(\frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}\right) - b^2}{a\,.\,AD}$$

$$= \frac{c^2 - b^2}{2a\,.\,AD},$$

and
$$\sin\theta = \frac{AH}{AD} = \frac{c\sin B}{AD}$$
$$= \frac{2\Delta}{a \cdot AD},$$

it follows that
$$\cot\theta = \frac{c^2 - b^2}{4\Delta}.$$

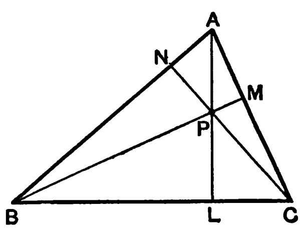
The Pedal Triangle.

150. The pedal triangle LMN is obtained by joining the feet of the perpendiculars from the angular points of a triangle to the opposite sides.

The point of intersection, P, of these perpendiculars is called the *Orthocentre.*

151. *To find the distances of the orthocentre from the angles and sides of the triangle.*

$$PL = BL\tan PBL = AB\cos B\cot MCB$$
$$= c\cos B\cot C$$
$$= \frac{c}{\sin C}\cos B\cos C$$
$$= 2R\cos B\cos C.$$

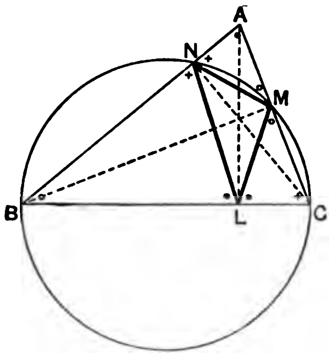


Similarly,
$$PM = 2R\cos C\cos A,$$
$$PN = 2R\cos A\cos B.$$
$$PA = AM\sec PAM$$
$$= AB\cos A\operatorname{cosec} ACL$$
$$= \frac{c}{\sin C}\cos A$$
$$= 2R\cos A.$$

Similarly,
$$PB = 2R\cos B,$$
$$PC = 2R\cos C.$$

152. *To find the angles and sides of the Pedal Triangle.*

Since BNMC is concyclic



$\therefore\ A\hat{N}M = 180° - BNM = C,$

$A\hat{M}N = 180° - NMC = B.$

Similarly

$B\hat{N}L = C$ etc.

$\therefore\ M\hat{N}L = 180° - 2C,$

$N\hat{L}M = 180° - 2A,$

$L\hat{M}N = 180° - 2B.$

Since BC ($= a$) is the diameter of the circle through BNMC

$\therefore\ MN = a \sin NBM$ (Art. 73)

$= a \cos A = 2R \sin A \cos A = R \sin 2A.$

Similarly $NL = b \cos B = R \sin 2B,$

$LM = c \cos C = R \sin 2C.$

153. *To find the area of the Pedal Triangle and the radius of its circum-circle.*

Area of $LMN = \frac{1}{2} NL . NM \sin LNM$

$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C.$

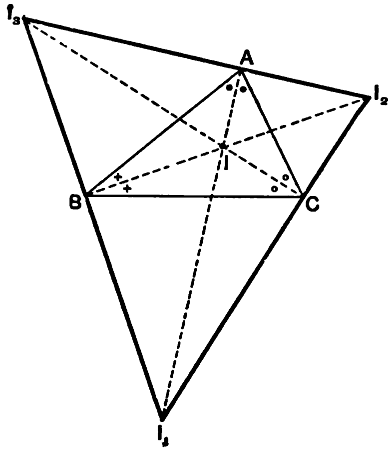
Radius of circum-circle

$$= \frac{\text{any side}}{2 \sin (\text{opposite angle})} = \frac{MN}{2 \sin MLN}$$

$$= \frac{R \sin 2A}{2 \sin 2A} = \frac{R}{2}.$$

The Ex-central Triangle.

154. If I_1, I_2, I_3 are the *e*-centres of the triangle ABC, then $I_1I_2I_3$ is called the Ex-central Triangle of ABC. By Geometry, AII_1, BII_2, CII_3, are straight lines as are also I_2AI_3, I_3BI_1, I_1CI_2, the first three being respectively perpendicular to the second three.



Thus ABC is the pedal triangle of $I_1I_2I_3$.

By making use of the results obtained for the Pedal Triangle we can thus obtain the properties of the Ex-central Triangle.

$$B\hat{A}C = 180° - 2I_3\hat{I_1}I_2$$

$$\therefore\ I_3\hat{I_1}I_2 = 90° - \frac{A}{2}.$$

Similarly

$$I_1\hat{I_2}I_3 = 90° - \frac{B}{2}$$

$$I_2\hat{I_3}I_1 = 90° - \frac{C}{2},$$

$$BC = I_2I_3 \cos I_3I_1I_2$$

$$= I_2I_3 \cos\left(90° - \frac{A}{2}\right),$$

$$\therefore\ I_2I_3 = \frac{a}{\sin\frac{A}{2}}.$$

Similarly $$I_3I_1 = \frac{b}{\sin\frac{B}{2}}$$

$$I_1I_2 = \frac{c}{\sin\frac{C}{2}}.$$

The values may easily be proved equal to

$$4R\cos\frac{A}{2},\quad 4R\cos\frac{B}{2},\quad 4R\cos\frac{C}{2}.$$

155. Area of $I_1I_2I_3 = \frac{1}{2} I_1I_3 . I_1I_2 \sin I_3I_1I_2$

$$= \tfrac{1}{2} . 16R^2 \cos\frac{B}{2}\cos\frac{C}{2}\sin\left(90° - \frac{A}{2}\right)$$

$$= 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Radius of circum-circle of $ABC = \frac{1}{2}$ radius of circum-circle of $I_1I_2I_3$.

$\therefore$ Rad. of circum-circle of $I_1I_2I_3 = 2R$.

The Bisectors of the Angles.

156. Let AK and AK′ be the bisectors of internal angle BAC and its supplement respectively.

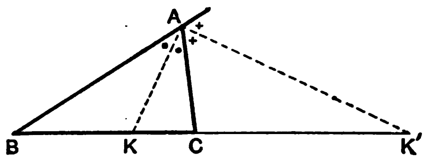
$$\frac{BK}{KC} = \frac{BA}{AC} = \frac{c}{b},$$

$$\therefore \frac{BK}{BK + KC} = \frac{c}{b+c},$$

$$\therefore BK = \frac{ac}{b+c}.$$

Similarly $$\mathsf{KC} = \frac{ab}{b+c}$$

$$\frac{\mathsf{BK'}}{\mathsf{K'C}} = \frac{\mathsf{BA}}{\mathsf{AC}} = \frac{c}{b},$$



$$\therefore \frac{\mathsf{BK'}}{\mathsf{BK'} - \mathsf{K'C}} = \frac{c}{c-b},$$

$$\therefore \mathsf{BK'} = \frac{ac}{c-b}.$$

Similarly $$\mathsf{CK'} = \frac{ab}{c-b}.$$

157. *To find the lengths of the bisectors.*

$$\triangle \mathsf{ABK} + \triangle \mathsf{AKC} = \triangle \mathsf{ABC}.$$

$$\therefore \tfrac{1}{2}\mathsf{AB}.\mathsf{AK}\sin\frac{\mathsf{A}}{2} + \tfrac{1}{2}\mathsf{AK}.\mathsf{AC}\sin\frac{\mathsf{A}}{2} = \tfrac{1}{2}\mathsf{AB}.\mathsf{AC}\sin\mathsf{A}$$

$$\mathsf{AK}(c+b)\sin\frac{\mathsf{A}}{2} = bc\sin\mathsf{A}$$

$$\mathsf{AK} = \frac{2bc}{b+c}\cos\frac{\mathsf{A}}{2}.$$

Similarly $$\triangle \mathsf{ABK'} - \triangle \mathsf{ACK'} = \triangle \mathsf{ABC}.$$

$$\tfrac{1}{2}\mathsf{AB}.\mathsf{AK'}\sin\mathsf{BAK'} - \tfrac{1}{2}\mathsf{AC}.\mathsf{AK'}\sin\mathsf{CAK'} = \tfrac{1}{2}\mathsf{AB}.\mathsf{AC}\sin\mathsf{A}$$

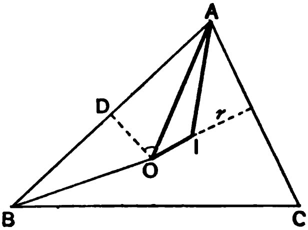
$$\mathsf{AK'}(c-b)\cos\frac{\mathsf{A}}{2} = bc\sin\mathsf{A},$$

$$\therefore \mathsf{AK'} = \frac{2bc}{c-b}\sin\frac{\mathsf{A}}{2}.$$

[These results have previously been proved in Art. 84.]

158. *To find the distance between the in-centre and the circum-centre.*

If I is the in-centre and O the circum-centre



$$I\hat{A}D = \frac{A}{2}$$

$$O\hat{A}D = 90° - AOD = 90° - C;$$

$$\therefore\ I\hat{A}O = \frac{A}{2} - 90° + C$$

$$= \frac{A - (A + B + C) + 2C}{2}$$

$$= \frac{C - B}{2}$$

$$AO = R$$

$$AI = \frac{r}{\sin\frac{A}{2}} = 4R \sin\frac{B}{2}\sin\frac{C}{2} \text{ (Art. 143).}$$

Therefore from the triangle IOA,

$$OI^2 = AO^2 + AI^2 - 2AO \,.\, AI \cos OAI$$

$$= R^2 + 16R^2 \sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2 \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$$= R^2 + 8R^2 \sin\frac{B}{2}\sin\frac{C}{2}\left[2\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{C}{2}\cos\frac{B}{2} - \sin\frac{C}{2}\sin\frac{B}{2}\right]$$

$$= R^2 - 8R^2 \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= R^2\left[1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}\right]$$

$$= R^2 - 2R \,.\, 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= R^2 - 2Rr.$$

159. Similarly if I_1, I_2, I_3 are the *e*-centres, we have

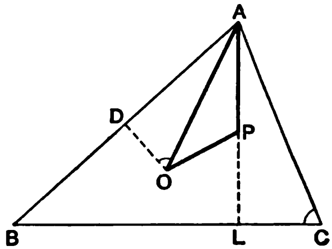
$$OI_1^2 = R^2 + 2Rr_1$$

$$OI_2^2 = R^2 + 2Rr_2$$

$$OI_3^2 = R^2 + 2Rr_3.$$

160. *To find the distance between the circum-centre and the orthocentre.*

Let O be the circum-centre and P the orthocentre.



$$O\hat{A}D = 90° - AOD = 90° - C$$

$$P\hat{A}D = 90° - ABL = 90° - B,$$

$$\therefore P\hat{A}O = P\hat{A}D - O\hat{A}D$$
$$= C - B$$

$$AO = R$$

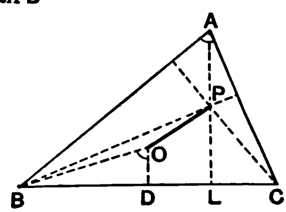
$$AP = 2R\cos A \text{ (Art. 151)}.$$

From the triangle OAP,

$$\begin{aligned} OP^2 &= AO^2 + AP^2 - 2AO \,.\, AP\cos OAP \\ &= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B) \\ &= R^2 + 4R^2\cos A[\cos A - \cos(C - B)] \\ &= R^2 + 4R^2\cos A[-\cos(C + B) - \cos(C - B)] \\ &= R^2 - 8R^2\cos A\cos B\cos C \\ &= R^2[1 - 8\cos A\cos B\cos C]. \end{aligned}$$

161. Ex. 1. Prove that the line joining O and P makes with BC an angle θ, where

$$\tan\theta = \frac{3 - \tan B\tan C}{\tan C - \tan B}.$$



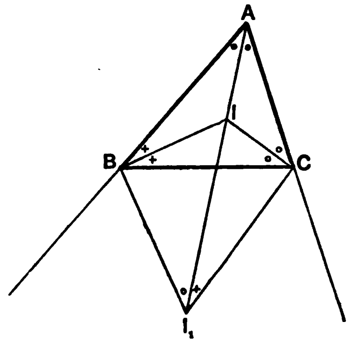
$$PL = c\cos B\cot C \text{ (Art. 151)},$$

$$OD = BD\cot A = \frac{a}{2}\cot A,$$

$$\begin{aligned} DL &= \tfrac{1}{2}[(BD + DL) - (CD - DL)] \\ &= \tfrac{1}{2}(BL - CL) \\ &= \tfrac{1}{2}(c\cos B - b\cos C), \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{PL - OD}{DL} = \frac{2c\cos B\cot C - a\cot A}{c\cos B - b\cos C} \\ &= \frac{2\cos B\cos C - \cos A}{\sin C\cos B - \sin B\cos C} \\ &= \frac{3\cos B\cos C - \sin B\sin C}{\sin C\cos B - \sin B\cos C} \quad [\text{since } \cos A = -\cos(B + C)] \\ &= \frac{3 - \tan B\tan C}{\tan C - \tan B}. \end{aligned}$$

Ex. 2. Prove that $\dfrac{II_1^2}{r_1 - r} = 4R.$



$B\hat{I_1}C = 90° - \dfrac{A}{2}$. (Art. 154.)

$I\hat{B}I_1 = I\hat{C}I_1 = 90°$.

$\therefore$ IBI_1C is concyclic, II_1 being a diameter.

$\therefore$ $BC = II_1 \sin BI_1C$ (Art. 73)

$$= II_1 \cos \frac{A}{2}.$$

$$\therefore\ II_1 = \frac{a}{\cos \frac{A}{2}}.$$

Also
$$\sin \frac{A}{2} = \frac{r_1}{AI_1} = \frac{r}{AI} = \frac{r_1 - r}{II_1},$$

$$4R = \frac{2a}{\sin A} = \frac{a}{\cos \frac{A}{2}} \cdot \frac{1}{\sin \frac{A}{2}} = II_1 \cdot \frac{II_1}{r_1 - r}$$

$$= \frac{II_1^2}{r_1 - r}.$$

EXAMPLES XXXVI.

If I, I_1, I_2, I_3, O, P be the in-centre, *e*-centres, circum-centre, and orthocentre of a triangle ABC,

Prove that

1. $\dfrac{IA \cdot IB}{IC} = 4R \sin^2 \dfrac{C}{2}$.

2. $\dfrac{I_1A \cdot I_1B}{I_1C} = 4R \cos^2 \dfrac{C}{2}$.

3. $IA \cdot IB \cdot IC = 4\Delta R \tan \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}$.

4. $\dfrac{II_1}{I_2I_3} = \tan \dfrac{A}{2}$.

5. Area of $I_1I_2I_3 = \dfrac{abc}{2r}$.

6. $\dfrac{\text{Area of } I_2I_3I_1}{\text{Area of } I_3I_1I} = \dfrac{r_1}{r}$.

7. If β and γ are the angles the median through A makes with AB and AC, then $c \sin \beta = b \sin \gamma$.

8. $IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

9. The radius of the inscribed circle of the pedal triangle is

$$2R \cos A \cos B \cos C.$$

10. $IA^2 + I_1A^2 + I_2A^2 + I_3A^2 = 16R^2$.

11. $IO^2 + I_1O^2 + I_2O^2 + I_3O^2 = 12R^2$.

12. $a \cdot BP \cdot CP + b \cdot CP \cdot AP + c \cdot AP \cdot BP = abc$.

13. Area of triangle

$IOP = -2R^2 \sin \frac{1}{2}(B - C) \sin \frac{1}{2}(C - A) \sin \frac{1}{2}(A - B)$.

14. $\Delta = r^2 \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C$.

15. $IA \cdot IB \cdot IC = 4Rr^2$.

16. If x, y, z are the perpendiculars from O to the sides of the triangle

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

17. $\dfrac{IA}{I_1A} + \dfrac{IB}{I_2B} + \dfrac{IC}{I_3C} = 1$.

18. If the perpendiculars AL, BM, CN from the angular points to the opposite sides, meet the circum-circle again in L′, M′, N′,

$$\frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4.$$

19. If the line IO makes an angle θ with BC,

$$\cot \theta = \frac{\sin B - \sin C}{\cos B + \cos C - 1}.$$

20. If the bisectors of the angles B and C meet the opposite sides in E, F, and the line EF makes an angle ϕ with BC,

$$\tan\phi = \frac{(b-c)\sin A}{(a+b)\cos C + (a+c)\cos B} = \frac{\sin B - \sin C}{\cos B + \cos C + 1}.$$

21. $$\frac{r_1 r_2 r_3}{r^3} = \frac{(a+b+c)^3}{(b+c-a)(c+a-b)(a+b-c)}.$$

22. If AL, BM, CN are the perpendiculars from the angular points to the opposite sides,

(i) the perimeter of LMN $= 4R \sin A \sin B \sin C$,

(ii) $$\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

23. Two circles are described with centres at the corners A, B of an acute-angled triangle ABC, so as to touch the sides BC, CA respectively. Prove that the angle θ at which the circles cut is given by

$$\cos\theta = \tfrac{1}{2}\cos C\,(\cot A \cot B + 1).$$

24. Prove that the diameter of the circum-circle through A is divided by BC in the ratio of $\tan B \tan C : 1$.

25. Perpendiculars from A, B, C on the opposite sides meet the circum-circle again in D, E, F. Prove that the ratio of the area of triangle DEF to that of ABC is $8 \cos A \cos B \cos C$.

26. The inscribed circle touches BC at D and the perpendicular from A on BC meets BC in E. Prove that

$$DE = \frac{(b-c)(b+c-a)}{2a}.$$

27. If AD is drawn perpendicular to BC and if ρ_1, ρ_2 denote the radii of the inscribed circles of the triangles ABD, ACD, show that

$$\frac{\cot B}{\rho_1} + \frac{\cot C}{\rho_2} = (\cot B + \cot C)\left(\frac{1}{r} + \frac{2}{a}\right).$$

28. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides, and Δ_1, Δ_2, Δ_3 corresponding areas for the escribed circles

$$s\Delta_0 = (s-a)\Delta_1 = (s-b)\Delta_2 = (s-c)\Delta_3.$$

29. If G is the intersection of the medians of a triangle ABC (area Δ), prove that

$$9AG^2 = 2\Delta(4 \cot A + \cot B + \cot C).$$

30. Prove that

$$OP^2 = 2R^2(\tfrac{3}{2} + \cos 2A + \cos 2B + \cos 2C).$$

31. If K is the centre of the circle circumscribing BPC, prove that

$$IK^2 = (R + r)^2 + r^2 - \frac{2\Delta^2}{r_2 r_3}.$$

32. If D, E, F are the mid-points of the sides of a triangle ABC, and D′, E′, F′ the feet of the perpendiculars from the vertices A, B, C on the opposite sides, prove that

$$\frac{a^2 \cos B \cos C}{EE' . FF'} - \frac{b^2 \cos C \cos A}{FF' . DD'} + \frac{c^2 \cos A \cos B}{DD' . EE'} = 4.$$

33. Prove that $(a + b + c)\, II_1 . II_2 . II_3 = 8Rabc$.

34. Given an isosceles triangle whose vertical angle is A, and base a, show that the diameter of the circle which cuts the sides of the triangle two and two in points which are at the opposite extremities of a diameter is

$$\frac{a}{2 - \cos A}.$$

35. If U is the centre of the nine-point circle of a triangle ABC, prove that

$$IU = \tfrac{1}{2}R - r.$$

36. On the base BC of a triangle ABC, a point V is taken such that $VC/VB = \sin 2C/\sin 2B$, whilst the line joining the circum-centre O and the orthocentre P meets BC at T. If VK be the perpendicular from V on OP, and if OP be bisected at I, then

$$4IK . IT = R^2.$$

37. Show that the orthocentre of a triangle lies on the inscribed circle if

$$\cos A \cos B \cos C = 4 \sin^2\frac{A}{2} \sin^2\frac{B}{2} \sin^2\frac{C}{2}.$$

38. If the line joining the circum-centre and in-centre of a triangle touches the escribed circle opposite the angle A, prove that

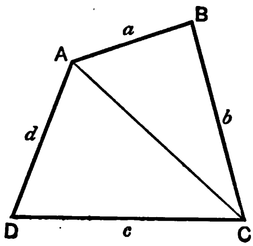
$$\Delta(r_2 - r_3) = r_1 r_2 r_3 \left(1 - \frac{2r}{R}\right)^{\frac{1}{2}}.$$

CHAPTER XV.

QUADRILATERALS AND POLYGONS.

162. *To find the area* (S) *of a quadrilateral.*

Let $\qquad \mathrm{B}+\mathrm{D}=2\alpha$ and $a+b+c+d=2s$.



$$\mathrm{AC}^2=a^2+b^2-2ab\cos\mathrm{B}$$
$$=c^2+d^2-2cd\cos\mathrm{D},$$

$\therefore\ a^2+b^2-c^2-d^2$

$$=2(ab\cos\mathrm{B}-cd\cos\mathrm{D})\ldots\ldots\text{(i)}.$$

Also

$$4\mathrm{S}=4\mathrm{ABC}+4\mathrm{ACD}$$
$$=2(ab\sin\mathrm{B}+cd\sin\mathrm{D})\ldots\ldots\text{(ii)}.$$

$\therefore$ squaring and adding,

$$16\mathrm{S}^2+(a^2+b^2-c^2-d^2)^2$$
$$=4[a^2b^2+c^2d^2-2abcd\cos(\mathrm{B}+\mathrm{D})]$$
$$=4[a^2b^2+c^2d^2-2abcd\cos 2\alpha]$$
$$=4[a^2b^2+c^2d^2-2abcd(2\cos^2\alpha-1)]$$
$$=4(ab+cd)^2-16abcd\cos^2\alpha;$$

$$\therefore\ 16\mathrm{S}^2=4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2-16abcd\cos^2\alpha$$
$$=[2(ab+cd)+(a^2+b^2-c^2-d^2)]$$
$$[2(ab+cd)-(a^2+b^2-c^2-d^2)]-16abcd\cos^2\alpha$$
$$=[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]-16abcd\cos^2\alpha$$
$$=(a+b+c-d)(a+b-c+d)(c+d+a-b)$$
$$(c+d-a+b)-16abcd\cos^2\alpha$$
$$=(2s-2d)(2s-2c)(2s-2b)(2s-2a)-16abcd\cos^2\alpha;$$

$$\therefore\ \mathbf{S^2=(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\alpha.}$$

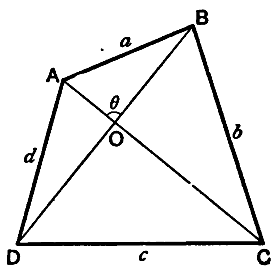
In the case of a cyclic quadrilateral,

$$B + D = 2\alpha = 180°,$$

$$\therefore \cos\alpha = 0.$$

Thus $$S^2 = (s-a)(s-b)(s-c)(s-d).$$

163. The area may also be found in terms of the diagonals and the angle between them.



$$\begin{aligned} 2S &= 2\triangle AOB + 2\triangle BOC + 2\triangle AOD + 2\triangle DOC \\ &= AO\,.\,OB\sin\theta + BO\,.\,OC\sin(\pi-\theta) \\ &\quad + AO\,.\,OD\sin(\pi-\theta) + DO\,.\,OC\sin\theta \\ &= AO\,.\,DB\sin\theta + BD\,.\,OC\sin\theta, \end{aligned}$$

$$\therefore S = \tfrac{1}{2}AC\,.\,DB\sin\theta.$$

164. In the case of a cyclic quadrilateral, since B and D are supplementary, equation (i), Art. 162, becomes

$$a^2 + b^2 - c^2 - d^2 = 2(ab+cd)\cos B,$$

i.e. $$\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

and from (ii) $$\sin B = \frac{2S}{ab+cd}.$$

165. *To find the diagonals and circum-radius* (R) *of a cyclic quadrilateral.*

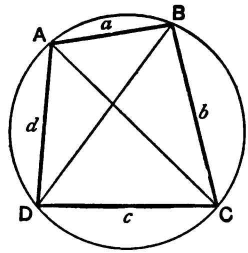
We have shown that $AC^2 = a^2 + b^2 - 2ab \cos B$.

Substituting for cos B from Art. 164 we have

$$AC^2 = a^2 + b^2 - \frac{ab(a^2 + b^2 - c^2 - d^2)}{ab + cd}$$

$$= \frac{cd(a^2 + b^2) + ab(c^2 + d^2)}{ab + cd}$$

$$= \frac{(ac + bd)(ad + bc)}{ab + cd}.$$



Similarly

$$BD^2 = \frac{(ab + cd)(ac + bd)}{ad + bc}.$$

The circle circumscribing ABCD also circumscribes the triangle ABC;

$$\therefore\ R = \frac{AC}{2 \sin B} = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}} \cdot \frac{ab + cd}{4S} \quad \text{(Art. 164)}$$

$$= \frac{1}{4S} \sqrt{(ab + cd)(ac + bd)(ad + bc)}.$$

166. Ex. 1. Find the area of a cyclic quadrilateral when the sides are 4, 5, 7, 8 centimetres respectively.

$$s = \frac{4 + 5 + 7 + 8}{2} = 12.$$

$$\therefore\ S = \sqrt{8 \cdot 7 \cdot 5 \cdot 4} \text{ sq. cms.}$$

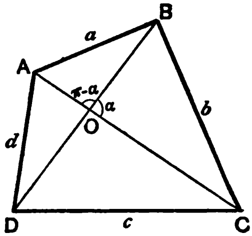
$$= 4\sqrt{70} \text{ sq. cms.}$$

$$= 33{\cdot}46 \text{ sq. cms.}$$

(correct to the nearest sq. millimetre).

Ex. 2. If a, b, c, d are the sides of a quadrilateral and α the angle opposite b between the diagonals, prove that the area of the quadrilateral is

$$\tfrac{1}{4}(a^2+c^2-b^2-d^2)\tan\alpha.$$



$$2OC \,.\, OB\cos\alpha = OC^2+OB^2-b^2$$
$$2OA \,.\, OB\cos(\pi-\alpha) = OA^2+OB^2-a^2.$$

$\therefore$ subtracting

$$2AC \,.\, OB\cos\alpha = OC^2-OA^2-b^2+a^2 \;\ldots\ldots\text{(i)}.$$

Also

$$2OA \,.\, OD\cos\alpha = OA^2+OD^2-d^2$$
$$2OC \,.\, OD\cos(\pi-\alpha) = OC^2+OD^2-c^2.$$

Subtracting, $\quad 2AC \,.\, OD\cos\alpha = OA^2-OC^2-d^2+c^2 \;\ldots\ldots\ldots\ldots\text{(ii)}.$

$\therefore$ adding (i) and (ii),

$$2AC \,.\, BD\cos\alpha = a^2+c^2-b^2-d^2.$$

Now $\quad 2AC \,.\, BD\sin\alpha = 4S, \qquad$ (Art. 163)

$$\therefore\ \tan\alpha = \frac{4S}{a^2+c^2-b^2-d^2},$$

or $\quad S = \tfrac{1}{4}(a^2+c^2-b^2-d^2)\tan\alpha.$

EXAMPLES XXXVII.

1. If the sides of a cyclic quadrilateral are 2, 4, 8, 6 centimetres respectively, find the area. [Answer to the nearest sq. millimetre.]

2. Find the lengths of the diagonals of a cyclic quadrilateral, if the sides taken in order are 3, 5, 7, 10 centimetres respectively.

Also find the radius of the circumscribing circle. (Answer to the nearest millimetre.)

3. If 2α is the sum of two opposite angles of a quadrilateral which has a circle inscribed in it, prove that the area is

$$\sqrt{abcd}\sin\alpha.$$

4. If a circle can be inscribed in a cyclic quadrilateral, prove that the area of the quadrilateral is $\sqrt{abcd}$, and the radius of the circle

$$2\sqrt{abcd}/(a+b+c+d).$$

5. If a circle can be inscribed in a quadrilateral, prove that the area of the quadrilateral is

$$\tfrac{1}{2}\left[x^2y^2-(ac-bd)^2\right]^{\frac{1}{2}}$$

where x and y are the diagonals.

6. The area of any quadrilateral is

$$\tfrac{1}{4}\left[4x^2y^2-(b^2+d^2\div a^2-c^2)^2\right]^{\frac{1}{2}}$$

where x and y are the diagonals.

7. If ABCD is a cyclic quadrilateral, prove that

$$(s-c)(s-d)\tan^2\frac{\mathrm{B}}{2}=(s-a)(s-b).$$

8. If θ is the angle between the diagonals of a cyclic quadrilateral, prove that

$$(ac+bd)\sin\theta=2\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

$$2(ac+bd)\cos\theta=(a^2+c^2)\sim(b^2+d^2).$$

9. ABCD is a cyclic quadrilateral, the circle having unit radius; α, β, γ are the angles subtended by AB, BC, CD at the circumference; prove that

$$\text{area of ABCD}=2\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta).$$

10. If 2α is the sum of two opposite angles, ϕ the angle between the diagonals, and the quadrilateral such that a circle can be inscribed in it, prove that

$$\tan^2\phi=\frac{4abcd\sin^2\alpha}{(ac-bd)^2}.$$

11. A quadrilateral is formed of four jointed rods of lengths a, b, c, d. If the area of the quadrilateral when the angle between a, b is a right angle is equal to the area when the angle between c, d is a right angle, show that either $ab=cd$, or

$$a^2+b^2=c^2+d^2.$$

12. Show that if a, b are adjacent sides of a parallelogram, α, ϕ the acute angles between the sides and between the diagonals respectively, then

$$\frac{a}{b}\sin\phi=\sin\alpha\cos\phi\pm\sqrt{1-\cos^2\alpha\cos^2\phi}.$$

13. If equilateral triangles are described on the sides of a quadrilateral outwards, and their corners joined in succession to form another quadrilateral, prove that the sum of the squares of its sides is

$$3(a^2+b^2+c^2+d^2)+4\sqrt{3}S-x^2-y^2$$

where x and y are the diagonals of the original quadrilateral.

14. If x and y are the diagonals of a quadrilateral and θ the sum of two opposite angles, prove that

$$x^2y^2=a^2c^2+b^2d^2-2abcd\cos\theta.$$

15. If a circle can be described about a quadrilateral, the ratio of the tangents drawn to the circle from the intersections of opposite sides is

$$\frac{a^2-c^2}{b^2-d^2}\sqrt{\frac{bd}{ac}}.$$

16. If it is possible to draw two circles, one touching AB, BC, CD, the other touching CD, DA, AB, and the two circles touching one another, prove that

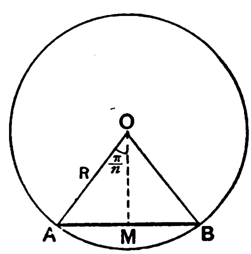
$$(a-b+c-d)\sin\frac{\mathrm{A+D}}{2}=4\sqrt{bd\sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\sin\frac{\mathrm{D}}{2}}.$$

REGULAR POLYGONS.

167. *To find the radius and area of a regular polygon of n sides inscribed in a circle.*

Let AB $(=a)$ be one of the sides of the polygon and O the centre of the circumscribing circle.

Draw OM perpendicular to AB.



$$\mathrm{A\hat{O}M}=\tfrac{1}{2}\mathrm{A\hat{O}B}$$
$$=\frac{1}{2}\cdot\frac{2\pi}{n}$$
$$=\frac{\pi}{n}$$
$$\mathrm{R}=\mathrm{AM}\operatorname{cosec}\mathrm{AOM}$$
$$=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\;\ldots\ldots\;\text{(i)}.$$

Area of polygon $= n \times$ area of AOB

$$= \frac{n}{2} \text{AB} \times \text{OM} = \frac{na}{2} \times \frac{a}{2} \cot \frac{\pi}{n}$$

$$= \frac{na^2}{4} \cot \frac{\pi}{n}.$$

By substituting for a from (i), this value becomes

$$\tfrac{1}{2} n \text{R}^2 \sin \frac{2\pi}{n}.$$

168. *To find the radius and area of a regular polygon of n sides circumscribed about a circle.*

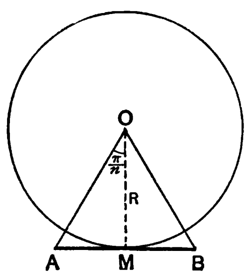
Let AB $(= a)$ be one side of the polygon, touching the circle at M.

Join OA, OB, OM.

$$\text{A}\hat{\text{O}}\text{M} = \tfrac{1}{2} \text{A}\hat{\text{O}}\text{B} = \frac{\pi}{n},$$

$$\text{R} = \text{AM} \cot \frac{\pi}{n}$$

$$= \frac{a}{2} \cot \frac{\pi}{n} \quad \text{......(ii).}$$



Area of polygon

$$= n \times \text{area of AOB}$$

$$= \frac{n}{2} \text{AB} \times \text{OM} = \frac{na}{2} \times \frac{a}{2} \cot \frac{\pi}{n}$$

$$= \frac{na^2}{4} \cot \frac{\pi}{n}.$$

By substituting for a from (ii), this value becomes

$$n\text{R}^2 \tan \frac{\pi}{n}.$$

169. Ex. If the length of one side of a regular pentagon is 5 centimetres, find its area, and the radius of the circumscribing circle.

$$R = \frac{5}{2}\operatorname{cosec}\frac{\pi}{5} = \frac{5}{2}\operatorname{cosec} 36^\circ,$$

$$= \frac{5}{2} \times 1{\cdot}7013 = 4{\cdot}2533 \text{ cms.}$$

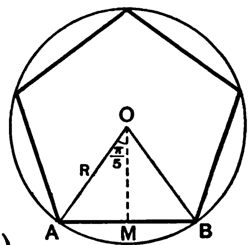
Area $= 5 \times$ area of AOB

$$= 5 \times \frac{5}{2}\,\text{OM}$$

$$= \frac{25}{2} \times \frac{5}{2}\cot 36^\circ$$

$$= \frac{125}{4} \times 1{\cdot}3764 \text{ sq. cms.}$$

$= 43$ sq. cms. (to the nearest sq. cm.)



EXAMPLES XXXVIII.

1. If the length of the side of a regular hexagon is 10 centimetres, find the radius of the inscribed circle and the area of the hexagon to the nearest sq. millimetre.

2. Find the perimeter of a regular octagon which surrounds a circle of radius 2 feet. (Answer to ·001 of a foot.)

3. Find the length of the side of a regular hexagon inscribed in a circle of radius 5 centimetres.

4. Find the area of a regular decagon inscribed in a circle of 6 inches radius. (Answer to $\frac{1}{100}$ of a sq. inch.)

5. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

6. Show that the areas of the inscribed and circumscribed circles of a regular hexagon are as 3 : 4.

7. Given that a regular hexagon has an area of 200 sq. centimetres, find the area of the circle inscribed in it. (Answer to the nearest sq. centimetre.)

8. If the area of a circle is 150 sq. inches, find the area of the regular pentagon described about it. (Answer to the nearest square inch.)

9. The area of a regular hexagon is 235 sq. centimetres. Find the length of one of the sides to the nearest millimetre.

10. Two regular polygons of n sides and $2n$ sides have the same perimeter, show that their areas are as $4\cos\frac{\pi}{n} : 1+\cos\frac{\pi}{n}$.

11. Two regular polygons of n sides are respectively circumscribed about and inscribed in a circle. Prove that their areas are as $\cos^2\frac{\pi}{n} : 1$.

12. Find the area enclosed by 200 hurdles placed so as to form a regular polygon of 200 sides, the length of each hurdle being 6 feet. (Answer to the nearest sq. foot.)

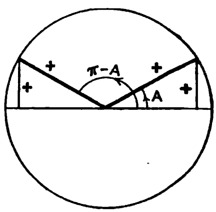
13. ABCDE is a regular pentagon. Show that if the distance of A from B or E be 34 inches, its distance from C or D will be 55 inches nearly.

CHAPTER XVI.

GENERAL VALUES OF ANGLES WHICH HAVE THE SAME SINE, COSINE, ETC.

170. THE only angles in the first four quadrants which have the *same sine* as A are from the figure

$$\mathsf{A} \text{ and } \pi - \mathsf{A}.$$



Now the addition or subtraction of any multiple of 2π makes no difference to the trigonometrical ratios of an angle.

Hence all the angles which have the same *sine* as A are found in the formulae

$$2r\pi + \mathsf{A}$$

and $$2r\pi + \pi - \mathsf{A}, \quad \textit{i.e.} \quad (2r+1)\pi - \mathsf{A},$$

r being any integer positive or negative; and all these angles are included in the formula

$$n\pi + (-1)^n \mathsf{A},$$

n being any integer positive or negative.

This is also the formula for all angles which have the same *cosecant* as A.

Hence the general solution of

$$\sin a\theta = \sin b\mathsf{A}, \quad \text{or} \quad \operatorname{cosec} a\theta = \operatorname{cosec} b\mathsf{A},$$

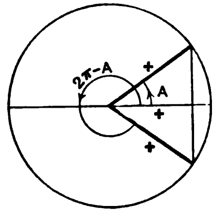
a and b being any constants, is

$$a\theta = n\pi + (-1)^n b\mathsf{A},$$

i.e. $$\theta = \frac{n\pi}{a} + (-1)^n \frac{b}{a}\mathsf{A}.$$

171. The only angles in the first four quadrants which have the *same cosine* as A are

$$A \text{ and } 2\pi - A.$$



Hence, as above, all the angles having the same *cosine* as A are included in the formulae

$$2r\pi + A \text{ and } 2r\pi + 2\pi - A,$$

r being any integer positive or negative; all these angles are included in the formula

$$2n\pi \pm A,$$

n being any integer positive or negative.

This is also the formula for all angles which have the same *secant* as A.

Hence the general solution of

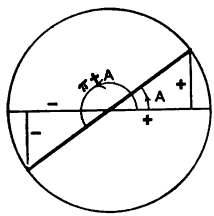
$$\cos a\theta = \cos bA \quad \text{or} \quad \sec a\theta = \sec bA,$$

is
$$a\theta = 2n\pi \pm bA,$$

i.e.
$$\theta = \frac{2n\pi}{a} \pm \frac{b}{a}A.$$

172. The only angles in the first four quadrants which have the *same tangent* as A are

$$A \text{ and } \pi + A.$$



Hence, as above, all the angles having the same *tangent* as A are included in the formulae

$$2r\pi + A \text{ and } 2r\pi + \pi + A,$$

r being any integer positive or negative; all these angles are included in the formula

$$n\pi + A,$$

n being any integer positive or negative.

This is also the formula for all angles which have the same *cotangent* as A.

Hence the general solution of

$$\tan a\theta = \tan b\text{A} \quad \text{or} \quad \cot a\theta = \cot b\text{A}$$

is

$$a\theta = n\pi + b\text{A},$$

i.e.

$$\theta = \frac{n\pi}{a} + \frac{b}{a}\text{A}.$$

173. It is interesting to notice that when an equation involves a square, the solution is always

$$n\pi \pm \text{A}.$$

For if

$$\sin^2\theta = \sin^2\text{A}$$

then

$$1 - \sin^2\theta = 1 - \sin^2\text{A},$$

$$\therefore \cos^2\theta = \cos^2\text{A};$$

$$\therefore \tan^2\theta = \tan^2\text{A};$$

or if

$$\cos^2\theta = \cos^2\text{A}$$

then

$$\sin^2\theta = \sin^2\text{A},$$

$$\therefore \tan^2\theta = \tan^2\text{A};$$

and thus every such equation is equivalent to

$$\tan^2\theta = \tan^2\text{A},$$

$$\therefore \tan\theta = \tan\text{A} \text{ or } \tan(-\text{A}),$$

$$\therefore \theta = n\pi \pm \text{A}.$$

ILLUSTRATIVE EXAMPLES.

174. Ex. 1. Solve

$$3\sin 7\theta - 2\sin 4\theta + 3\sin\theta = 0.$$

$$3(\sin 7\theta + \sin\theta) = 2\sin 4\theta,$$

$$6\sin 4\theta\cos 3\theta = 2\sin 4\theta.$$

$\therefore$ either $$\sin 4\theta = 0 = \sin\frac{\pi}{2},$$

i.e. $$4\theta = n\pi + (-1)^n\frac{\pi}{2},$$

i.e. $$\theta = \frac{n\pi}{4} + (-1)^n\frac{\pi}{8},$$

or $$\cos 3\theta = \tfrac{1}{3} = \cdot\dot{3} = \cos 70^\circ\, 32',$$

i.e. $$3\theta = 2n\pi \pm 70^\circ 32'$$

$$\theta = \frac{2n\pi}{3} \pm 23^\circ\, 30'\tfrac{2}{3}.$$

Ex. 2. Solve

$$\cos a\theta = \sin b\theta.$$

$$\cos a\theta = \cos\left(\frac{\pi}{2} - b\theta\right),$$

$$a\theta = 2n\pi \pm \left(\frac{\pi}{2} - b\theta\right),$$

$$\theta = \frac{(4n+1)\pi}{2(a+b)},$$

or $$= \frac{(4n-1)\pi}{2(a-b)},$$

we might have started

$$\sin\left(\frac{\pi}{2} - a\theta\right) = \sin b\theta \text{ etc.}$$

Ex. 3. Solve

$$7\cos\theta + \sin\theta = 2.$$

1*st method.* Change θ into $\frac{\theta}{2}$, thus

$$7\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right).$$

Dividing by $\cos^2\frac{\theta}{2}$

$$7\left(1 - \tan^2\frac{\theta}{2}\right) + 2\tan\frac{\theta}{2} = 2\left(1 + \tan^2\frac{\theta}{2}\right),$$

$$9\tan^2\frac{\theta}{2} - 2\tan\frac{\theta}{2} - 5 = 0.$$

Solving $\tan\frac{\theta}{2} = \cdot 8647$ or $-\cdot 6425$,

$$\therefore \tan\frac{\theta}{2} = \tan 40^\circ 51' \text{ or } = \tan(-32^\circ 43'),$$

$$\therefore \frac{\theta}{2} = n\pi + 40^\circ 51' \text{ or } = n\pi - 32^\circ 43',$$

i.e. $\theta = 2n\pi + 81^\circ 42'$ or $2n\pi - 65^\circ 26'$.

2nd method. Divide by the sum of the squares of the coefficients of $\cos\theta$ and $\sin\theta$.

$$\frac{7}{\sqrt{7^2+1^2}}\cos\theta + \frac{1}{\sqrt{7^2+1^2}}\sin\theta = \frac{2}{\sqrt{7^2+1^2}}.$$

From tables $\frac{1}{7} = \tan 8^\circ 8'$,

$$\therefore \frac{7}{\sqrt{7^2+1^2}} = \cos 8^\circ 8'; \quad \frac{1}{\sqrt{7^2+1^2}} = \sin 8^\circ 8',$$

also $\frac{2}{\sqrt{7^2+1^2}} = \cdot 2828 = \cos 73^\circ 34'$,

$$\therefore \cos 8^\circ 8' . \cos\theta + \sin 8^\circ 8' . \sin\theta = \cos 73^\circ 34',$$

$$\cos(\theta - 8^\circ 8') = \cos 73^\circ 34',$$

$$\therefore \theta - 8^\circ 8' = 2n\pi \pm 73^\circ 34',$$

$$\theta = 2n\pi + 81^\circ 42' \text{ or } 2n\pi - 65^\circ 26'.$$

Examples on this method have generally been set so as to be done by known angles; thus

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2},$$

$$\therefore \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}},$$

$$\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta = \cos 45^\circ,$$

$$\cos(\theta - 30^\circ) = \cos 45^\circ,$$

$$\therefore \theta = 2n\pi \pm 45^\circ + 30^\circ = 2n\pi + \frac{5\pi}{12} \text{ or } 2n\pi - \frac{\pi}{12}.$$

Ex. 4. Solve

$$(1-\tan\theta)=(1-3\tan\theta)\cos^2\theta$$

$$=\frac{(1-3\tan\theta)}{1+\tan^2\theta},$$

$$\therefore\ 1-\tan\theta+\tan^2\theta-\tan^3\theta=1-3\tan\theta,$$

$\therefore$ either $\tan\theta=0;\ \ i.e.\ \ \theta=n\pi,$

or $\tan^2\theta-\tan\theta-2=0,$

$$\therefore\ \tan\theta=2\ \text{ or }\ \tan\theta=-1,$$

$$\therefore\ \tan\theta=\tan 63^\circ\,26', \qquad \text{or}\ \ \tan\theta=\tan\left(-\frac{\pi}{4}\right),$$

$$\therefore\ \theta=n\pi+63^\circ\,26', \qquad \text{or}\ \ \theta=n\pi-\frac{\pi}{4}.$$

Ex. 5. Solve

$$\sin^2\theta-\cos 2\theta=1\tfrac{1}{4}.$$

$$\sin^2\theta-(1-2\sin^2\theta)=1\tfrac{1}{4},$$

$$3\sin^2\theta=\tfrac{9}{4},$$

$$\sin^2\theta=\sin^2 60^\circ,$$

$$\therefore\ \theta=n\pi\pm 60^\circ.\quad \text{(Art. 173.)}$$

EXAMPLES XXXIX.

Find the general solution of

1. $\sin 2\theta=\frac{\sqrt{3}}{2}$.
2. $\cos 3\theta=\frac{1}{2}$.
3. $\tan 4\theta=1$.
4. $\sin 5\theta=\cdot 3502$.
5. $\cos 6\theta=\cdot 95$.
6. $\tan 7\theta=\cdot 7032$.
7. $\sin^2 3\theta=\frac{3}{4}$.
8. $\cos^2 3\theta=\frac{1}{4}$.
9. $\tan^2 3\theta=3$.
10. $\sin 2\theta=\sin\theta$.

11. $\cos 3\theta = \cos 2\theta$.
12. $\tan 4\theta = \tan 3\theta$.
13. $\sin^2 5\theta = \sin^2 \theta$.
14. $\cos^2 4\theta = \cos^2 3\theta$.
15. $\tan^2 3\theta = \tan^2 \theta$.
16. $\cos 3\theta = \sin 2\theta$.
17. $\sin 5\theta = \cos 3\theta$.
18. $\tan 7\theta = \cot 2\theta$.
19. $\sin 4\theta + \sin 2\theta = \sin 3\theta$.
20. $\cos \theta - \cos 7\theta = \sin 4\theta$.
21. $\sin 5\theta - \sin 3\theta = \frac{1}{2} \cos 4\theta$.
22. $\cos 6\theta + \cos 2\theta = (1{\cdot}4216) \cos 4\theta$.
23. $\sin 7\theta + \sin 5\theta + \sin 3\theta + \sin \theta = 0$.
24. $\cos 9\theta + \cos 7\theta - \sin 5\theta - \sin 3\theta = 0$.
25. $\sin 7\theta \sin 5\theta = \sin 3\theta \sin \theta$.
26. $\cos 9\theta \cos 7\theta = \cos 5\theta \cos 3\theta$.
27. $\sin 7\theta \cos \theta = \sin 5\theta \cos 3\theta$.
28. $\sin \theta \cos 3\theta = \sin 2\theta \cos 4\theta$.
29. $5 \cos \theta + 2 \sin \theta = 4$.
30. $8 \cos \theta + 3 \sin \theta = 5$.
31. $4 \cos \theta + 3 \sin \theta = 5$.
32. $7 \cos \theta + 2 \sin \theta = 7$.
33. $\sqrt{3} \sin \theta - \cos \theta = 1$.
34. $\sin \theta + \cos \theta = \sqrt{2}$.
35. $\sin \theta + \sqrt{3} \cos \theta = \sqrt{2}$.
36. $\sin \theta - 1 = \sqrt{3} \cos \theta$.
37. $\cos 2\theta = \cos \theta - \sin \theta$.
38. $\sin 6\theta \sin 2\theta = \frac{1}{2}$.
39. $4 \cos 3\theta + 3 \cos \theta = 0$.
40. $\tan 2\theta + 3 \cot \theta = 0$.
41. $2 \sin x - \sin 2x = 2 (1 + \cos x)^2$.
42. $\tan^2 \theta - 4 \sec \theta + 5 = 0$.
43. $\tan 2\theta = 8 \cos^2 \theta - \cot \theta$.
44. $\tan \left(\frac{\pi}{4} + \theta\right) = 3 \tan \left(\frac{\pi}{4} - \theta\right)$.
45. $\dfrac{(\sin 2\theta - \cos 2\theta)}{\sqrt{2}} = 2 \sin^2 \theta - 1$.

46. $\cot 3x - 3\cot 2x + \cot x = 0$.

47. $\sin\theta + 1 = \cos\theta + \tan\theta$.

48. $8\cot\theta = \sec^2\dfrac{\theta}{2} + \operatorname{cosec}^2\dfrac{\theta}{2}$.

49. $\tan x + \tan(x+\alpha) + \tan(x+\beta) = \tan x \tan(x+\alpha)\tan(x+\beta)$.

50. $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$.

51. $2\sin^2 x + 3\cos x = 0$.

52. $3(1-\cos x) = \sin^2 x(3 - 2\cos x)$.

53. $\sin\left(\dfrac{\pi}{4} + \dfrac{3\theta}{2}\right) = 2\sin\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

54. $\dfrac{\cos\alpha}{\cos 2x} + \dfrac{\sin\alpha}{\sin 2x} = 2$.

55. $\sin(\alpha + x) + \sin(\beta + x) = 0$.

CHAPTER XVII.

SUBMULTIPLE ANGLES.

To express the Trigonometrical Ratios of half an angle in terms of those of the whole angle.

175. Given $\cos\alpha = k$, find $\sin\frac{\alpha}{2}, \cos\frac{\alpha}{2}$.

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} = \frac{1+k}{2},$$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1-k}{2};$$

$$\therefore \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+k}{2}}; \quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-k}{2}}.$$

It will be noticed

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}; \quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the first quadrant,

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}; \quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the second quadrant,

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}; \quad \sin\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the third quadrant,

$$\cos\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{2}};\quad \sin\frac{\alpha}{2}=-\sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the fourth quadrant.

Considerations of the double value.

176. (i) *Arithmetical.*

Given $\cos\alpha = \cdot 6261.$

We have from Tables

$$\alpha = 51^\circ\, 14'$$

∴ also from Chap. VI,

$$\alpha = -51^\circ\, 14'$$
$$\alpha = 360^\circ - 51^\circ\, 14' = 308^\circ\, 46'$$
$$\alpha = 360^\circ + 51^\circ\, 14' = 411^\circ\, 14'$$

therefore

$$\cos\frac{\alpha}{2} = \cos\ 25^\circ\, 37' = \cdot 9017;\quad \sin\frac{\alpha}{2} = \sin\ 25^\circ\, 37' = \cdot 4324$$

$$\cos\frac{\alpha}{2} = \cos(-25^\circ\, 37') = \cdot 9017;\quad \sin\frac{\alpha}{2} = \sin(-25^\circ 37') = -\cdot 4324$$

$$\cos\frac{\alpha}{2} = \cos 154^\circ\, 23' = -\cdot 9017;\quad \sin\frac{\alpha}{2} = \sin(-154^\circ 23') = \cdot 4324$$

$$\cos\frac{\alpha}{2} = \cos 205^\circ\, 37' = -\cdot 9017;\quad \sin\frac{\alpha}{2} = \sin 205^\circ\, 37' = -\cdot 4324,$$

$$\text{i.e. } \cos\frac{\alpha}{2} = \pm\cdot 9017;\quad \sin\frac{\alpha}{2} = \pm\cdot 4324.$$

We shall now show that these results obtained from first principles are the same as those found from Art. 175.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} = \pm\sqrt{\frac{1+\cdot 6261}{2}} = \pm\cdot 9017$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} = \pm\sqrt{\frac{1-\cdot 6261}{2}} = \pm\cdot 4324.$$

177. (ii) *Algebraical.*

$$\cos\alpha = k = \cos A, \text{ suppose,}$$

where A is the smallest positive angle satisfying the equation,

then $$\alpha = 2n\pi \pm A.$$

$$\therefore\ \cos\frac{\alpha}{2} = \cos\left(n\pi \pm \frac{A}{2}\right);\quad \sin\frac{\alpha}{2} = \sin\left(n\pi \pm \frac{A}{2}\right),$$

(*a*) when n is even $= 2m$ suppose

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi \pm \frac{A}{2}\right) = \cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi \pm \frac{A}{2}\right) = \pm\sin\frac{A}{2},$$

(*b*) when n is odd $= 2m + 1$ suppose

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi + \pi \pm \frac{A}{2}\right) = -\cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi + \pi \pm \frac{A}{2}\right) = \mp\sin\frac{A}{2},$$

$\therefore$ for all values of n

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2};\quad \sin\frac{\alpha}{2} = \pm\sin\frac{A}{2}.$$

$$= \pm\sqrt{\frac{1+k}{2}}\qquad = \pm\sqrt{\frac{1-k}{2}}.$$

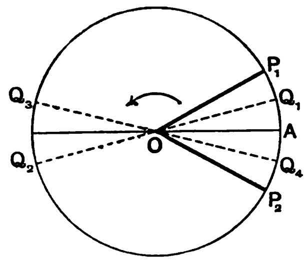
178. (iii) *Geometrical.*

$$\cos\alpha = k = \cos A, \text{ suppose,}$$

where A is the smallest positive angle satisfying the equation.

Let $$A\hat{O}P_1 = A,$$
$$A\hat{O}P_2 = 2\pi - A.$$

Let OP revolve positively so as to trace out α, and OQ revolve positively *half as fast*, so as to trace out $\frac{\alpha}{2}$.



Every time OP passes the positions OP_1, OP_2, α satisfies the equation $\cos\alpha = k$, and at these moments the position of OQ will afford us values of $\frac{\alpha}{2}$.

When OP is at OP_1 the 1st time, OQ is at OQ_1,
„ „ „ 2nd „ „ OQ_2,
„ „ „ 3rd „ „ OQ_1,
„ „ „ 4th „ „ OQ_2,
and so on.

Again,

when OP is at OP_2 the 1st time, OQ is at OQ_3,
„ „ „ 2nd „ „ OQ_4,
„ „ „ 3rd „ „ OQ_3,
„ „ „ 4th „ „ OQ_4,
and so on.

Thus

$$\begin{aligned}\cos\frac{\alpha}{2} &= \cos AOQ_1 = \cos\frac{A}{2}\\ &= \cos AOQ_2 = -\cos\frac{A}{2}\\ &= \cos AOQ_3 = -\cos\frac{A}{2}\\ &= \cos AOQ_4 = \cos\frac{A}{2}\end{aligned}$$

$$\begin{aligned}\sin\frac{\alpha}{2} &= \sin AOQ_1 = \sin\frac{A}{2}\\ &= \sin AOQ_2 = -\sin\frac{A}{2}\\ &= \sin AOQ_3 = \sin\frac{A}{2}\\ &= \sin AOQ_4 = -\sin\frac{A}{2}.\end{aligned}$$

Thus

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2} = \pm\sqrt{\frac{1+k}{2}}, \quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-k}{2}}.$$

179. Given $\sin\alpha = h$; find $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$.

$$\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2 = \cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 1+h,$$

$$\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2 = \cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 1-h,$$

$$\therefore\ \cos\frac{\alpha}{2}+\sin\frac{\alpha}{2} = \pm\sqrt{1+h},$$

$$\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2} = \pm\sqrt{1-h},$$

therefore

$$\cos\frac{\alpha}{2} = \frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2}, \qquad \sin\frac{\alpha}{2} = \frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2} \quad \ldots\text{(A)},$$

$$\text{or}\ \ \frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2}, \qquad \text{or}\ \ \frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2} \quad \ldots\text{(B)},$$

$$\text{or}\ -\frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2}, \qquad \text{or}\ -\frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2} \quad \ldots\text{(C)},$$

$$\text{or}\ -\frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2}, \qquad \text{or}\ -\frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2} \quad \ldots\text{(D)}.$$

180. Since

$$\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right) = \sqrt{2}\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right),$$

and

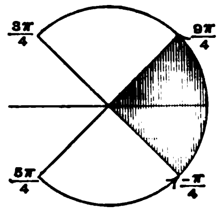
$$\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right) = -\sqrt{2}\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right),$$

we see that

(A) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is positive and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ negative;

i.e. when $\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is in first and second quadrants; and $\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ is in third and fourth quadrants;

i.e. when $\frac{\alpha}{2}$ lies between $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{5\pi}{4}$ and $\frac{9\pi}{4}$,



i.e. when $\frac{\alpha}{2}$ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, or generally when $\frac{\alpha}{2}$ lies between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$.

(B) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is positive and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ positive;

i.e. when $\frac{\alpha}{2}$ lies between $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$,

i.e. generally when $\frac{\alpha}{2}$ lies between $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$.

(C) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is negative and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ positive;

i.e. when $\frac{\alpha}{2}$ lies between $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$,

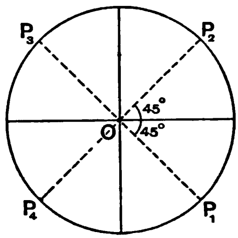
i.e. generally when $\frac{\alpha}{2}$ lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$.

(D) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is negative and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ negative;

i.e. when $\frac{\alpha}{2}$ lies between $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{5\pi}{4}$ and $\frac{9\pi}{4}$,

i.e. generally when $\frac{\alpha}{2}$ lies between $2n\pi+\frac{5\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$.

181. Thus from a figure



$$\cos\frac{\alpha}{2} = \frac{\sqrt{1+\sin\alpha}}{2} + \frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2} = \frac{\sqrt{1+\sin\alpha}}{2} - \frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_1OP_2,

$$\cos\frac{\alpha}{2} = \frac{\sqrt{1+\sin\alpha}}{2} - \frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2} = \frac{\sqrt{1+\sin\alpha}}{2} + \frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_2OP_3,

$$\cos\frac{\alpha}{2} = -\frac{\sqrt{1+\sin\alpha}}{2} - \frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2} = -\frac{\sqrt{1+\sin\alpha}}{2} + \frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_3OP_4,

$$\cos\frac{\alpha}{2} = -\frac{\sqrt{1+\sin\alpha}}{2} + \frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2} = -\frac{\sqrt{1+\sin\alpha}}{2} - \frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_4OP_1.

This article will be found very useful in working out examples.

Considerations of the quadruple value.

182. (i) *Arithmetical.*

Given $\sin\alpha = \cdot 7797.$

We have from Tables

$$\alpha = 51^\circ\, 14',$$

also from Chap. VI

$$\alpha = 180^\circ - 51^\circ\, 14' = 128^\circ\, 46',$$
$$\alpha = 360^\circ + 51^\circ\, 14' = 411^\circ\, 14',$$
$$\alpha = 540^\circ - 51^\circ\, 14' = 488^\circ\, 46'.$$

$$\begin{aligned} \cos\frac{\alpha}{2} &= \cos\ 25^\circ\,37' = \cdot 9017; & \sin\frac{\alpha}{2} &= \sin\ 25^\circ\,37' = \cdot 4324 \\ &= \cos\ 64^\circ\,23' = \cdot 4324; & &= \sin\ 64^\circ\,23' = \cdot 9017 \\ &= \cos 205^\circ\,37' = -\cdot 9017; & &= \sin 205^\circ\,37' = -\cdot 4324 \\ &= \cos 244^\circ\,23' = -\cdot 4324; & &= \sin 244^\circ\,23' = -\cdot 9017. \end{aligned}$$

We shall now show that these results obtained from first principles are the same as those found from Art. 179.

$$\frac{\sqrt{1+\cdot 7797}}{2} + \frac{\sqrt{1-\cdot 7797}}{2} = \cdot 9017,$$

$$\frac{\sqrt{1+\cdot 7797}}{2} - \frac{\sqrt{1-\cdot 7797}}{2} = \cdot 4324.$$

183. (ii) *Algebraical.*

$$\sin\alpha = h = \sin \mathsf{A}, \text{ suppose,}$$

where A is the smallest positive angle satisfying the equation.

Then $\alpha = n\pi + (-1)^n \mathsf{A}.$

$$\therefore\ \cos\frac{\alpha}{2} = \cos\left\{\frac{n\pi}{2} + (-1)^n\frac{\mathsf{A}}{2}\right\};$$

$$\sin\frac{\alpha}{2} = \sin\left\{\frac{n\pi}{2} + (-1)^n\frac{\mathsf{A}}{2}\right\};$$

(*a*) when n is of form $4m$

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi + \frac{A}{2}\right) = \cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi + \frac{A}{2}\right) = \sin\frac{A}{2}.$$

(*b*) when n is of form $4m+1$

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi + \frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi + \frac{\pi}{2} - \frac{A}{2}\right) = \cos\frac{A}{2}.$$

(*c*) when n is of form $4m+2$

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi + \pi + \frac{A}{2}\right) = -\cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi + \pi + \frac{A}{2}\right) = -\sin\frac{A}{2}.$$

(*d*) when n is of form $4m+3$

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi + \pi + \frac{\pi}{2} - \frac{A}{2}\right) = -\sin\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi + \pi + \frac{\pi}{2} - \frac{A}{2}\right) = -\cos\frac{A}{2}.$$

Therefore for all values of n

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2}, \qquad \sin\frac{\alpha}{2} = \pm\sin\frac{A}{2},$$

$$\text{or } \pm\sin\frac{A}{2}, \qquad \text{or } \pm\cos\frac{A}{2}.$$

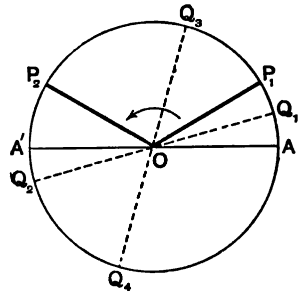
184. (iii) *Geometrical.*

$$\sin\alpha = h = \sin A, \text{ suppose,}$$

where A is the smallest positive angle satisfying the equation.

Let

$$AOP_1 = A;\quad AOP_2 = \pi - A.$$



Let OP revolve positively so as to trace out α, and OQ revolve positively *half as fast*, so as to trace out $\frac{\alpha}{2}$.

Every time OP passes the positions OP_1, OP_2, α satisfies the equation $\sin\alpha = h$; and at these moments the position of OQ will afford us values of $\frac{\alpha}{2}$.

When OP is at OP_1 the 1st time, OQ is at OQ_1,

„	„	„	2nd	„	„	OQ_2,
„	„	„	3rd	„	„	OQ_1,
„	„	„	4th	„	„	OQ_2,

and so on.

Again,

when OP is at OP_2 the 1st time, OQ is at OQ_3,

„	„	„	2nd	„	„	OQ_4,
„	„	„	3rd	„	„	OQ_3,
„	„	„	4th	„	„	OQ_4,

and so on.

Thus
$$\cos\frac{\alpha}{2} = \cos AOQ_1 = \cos\frac{A}{2}$$
$$= \cos AOQ_2 = -\cos\frac{A}{2}$$
$$= \cos AOQ_3 = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2}$$
$$= \cos AOQ_4 = -\sin\frac{A}{2}.$$

Hence

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2} = \pm\left(\frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2}\right)$$

$$\text{or } \pm\sin\frac{A}{2} \text{ or } \pm\left(\frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2}\right).$$

$$\begin{aligned}\sin\frac{\alpha}{2} &= \sin AOQ_1 = \sin\frac{A}{2}\\ &= \sin AOQ_2 = -\sin\frac{A}{2}\\ &= \sin AOQ_3 = \sin\left(\frac{\pi}{2}-\frac{A}{2}\right) = \cos\frac{A}{2}\\ &= \sin AOQ_4 = -\cos\frac{A}{2}.\end{aligned}$$

Hence

$$\sin\frac{\alpha}{2} = \pm\sin\frac{A}{2} = \pm\left(\frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2}\right),$$

$$\text{or } \pm\cos\frac{A}{2} \text{ or } \pm\left(\frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2}\right).$$

185. Given $\tan\alpha = k$, find $\tan\frac{\alpha}{2}$.

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = k,$$

$$\therefore\ \tan^2\frac{\alpha}{2} + \frac{2}{k}\tan\frac{\alpha}{2} - 1 = 0,$$

$$\therefore\ \tan\frac{\alpha}{2} = \frac{-1\pm\sqrt{1+k^2}}{k}.$$

Thus
$$\tan\frac{\alpha}{2} = \frac{-1+\sqrt{1+\tan^2\alpha}}{\tan\alpha},$$

when $\frac{\alpha}{2}$ lies in the first and third quadrants,

and $$\tan\frac{\alpha}{2}=\frac{-1-\sqrt{1+\tan^2\alpha}}{\tan\alpha},$$

when $\frac{\alpha}{2}$ lies in the second and fourth quadrants.

186. Algebraical consideration of the double value.

$$\tan\alpha=k=\tan \mathsf{A},\ \text{suppose},$$

where A is the smallest positive angle satisfying the equation, then $$\alpha=n\pi+\mathsf{A}.$$

Case (i) $$n \text{ even}=2m,$$
$$\frac{\alpha}{2}=m\pi+\frac{\mathsf{A}}{2};$$
$$\therefore\ \tan\frac{\alpha}{2}=\tan\frac{\mathsf{A}}{2}.$$

Case (ii) $$n \text{ odd}=2m+1,$$
$$\frac{\alpha}{2}=m\pi+\frac{\pi}{2}+\frac{\mathsf{A}}{2};$$
$$\therefore\ \tan\frac{\alpha}{2}=\tan\left(\frac{\pi}{2}+\frac{\mathsf{A}}{2}\right)=-\cot\frac{\mathsf{A}}{2}.$$

The arithmetical and geometrical considerations are left as an exercise for the student.

ILLUSTRATIVE EXAMPLES.

187. Ex. 1. Prove that

$$2\cos\frac{\theta}{2}=-\sqrt{1-\sin\theta}-\sqrt{1+\sin\theta},$$

when θ lies between 270° and 450°.

Since $\frac{\theta}{2}$ lies between 135° and 225°, *i.e.* in P_3OP_4 (Art. 181),

$$2\cos\frac{\theta}{2}=-\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}.$$

Ex. 2. Show that

$$\cos\frac{\pi}{16}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}};\ \sin\frac{\pi}{16}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}.$$

From Art. 175, $\cos\frac{\pi}{8}=+\sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2}}.$

$$\cos\frac{\pi}{16}=+\sqrt{\frac{1+\cos\frac{\pi}{8}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}.$$

Also

$$\sin\frac{\pi}{16}=+\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}.$$

Ex. 3. Given that $\sin 210^\circ=-\frac{1}{2}$, find the values of $\sin 105^\circ$ and $\cos 105^\circ$.

Since 105° lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, we have by Art. 181,

$$\cos 105^\circ=\frac{\sqrt{1+\sin 210^\circ}}{2}-\frac{\sqrt{1-\sin 210^\circ}}{2}$$

$$=\frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}}=\frac{1-\sqrt{3}}{2\sqrt{2}},$$

$$\sin 105^\circ=\frac{\sqrt{1+\sin 210^\circ}}{2}+\frac{\sqrt{1-\sin 210^\circ}}{2}$$

$$=\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}=\frac{1+\sqrt{3}}{2\sqrt{2}}.$$

EXAMPLES XL.

Prove that

1. $2\cos\frac{\theta}{2}=\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta},$

$2\sin\frac{\theta}{2}=\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta},$

when θ lies between 630° and 810° or between -810° and -630°.

2. $2\cos\frac{\theta}{2} = \sqrt{1+\sin\theta} - \sqrt{1-\sin\theta}$,

$$2\sin\frac{\theta}{2} = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta},$$

when θ lies between 810° and 990° or between $-630°$ and $-450°$ or between 90° and 270°.

3. $2\cos\frac{\theta}{2} = -\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta}$,

$$2\sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta},$$

when θ lies between 990° and 1170° or between $-450°$ and $-270°$ or between 270° and 450°.

4. $2\cos\frac{\theta}{2} = -\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$,

$$2\sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta},$$

when θ lies between 1170° and 1350° or between $-270°$ and $-90°$ or between 450° and 630°.

5. If $\theta = 200°, 400°, 600°, 800°, 1100°$, show that

$$\tan\theta = \frac{-1+\sqrt{1+\tan^2 2\theta}}{\tan 2\theta}.$$

6. If $\theta = 100°, 300°, 500°, 700°, 1000°$, show that

$$\tan\theta = \frac{-1-\sqrt{1+\tan^2 2\theta}}{\tan 2\theta}.$$

7. If $\cos A = \frac{7}{25}$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, A being between 270° and 360°.

8. If $\sin A = \frac{240}{289}$ and A lie between 270° and 450°, find the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

9. Having given that $\sin 260° = -0{\cdot}9848$, find the values of $\sin 130°$ and $\cos 130°$.

10. Find $\sin 115^\circ$ and $\cos 115^\circ$, given that

$$\cos 230^\circ = -0{\cdot}6428.$$

11. If $\tan 2\text{A} = \frac{24}{7}$, find the values of $\tan \text{A}$.

Prove that

12. $\sin \frac{\pi}{12} = \frac{1}{2}\sqrt{2 - \sqrt{3}}.$

13. $\cos \frac{\pi}{24} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{3}}}.$

14. $\sin \frac{\pi}{20} = \frac{1}{2}\sqrt{2 - \sqrt{\frac{1}{2}(5 + \sqrt{5})}} = \frac{1}{8}[\sqrt{10} + \sqrt{2} - 2\sqrt{5 - \sqrt{5}}].$

15. $\tan \frac{\pi}{8} = \sqrt{2} - 1.$

16. If $\cos 4\theta = a$, the possible values of $\tan \theta$ are the four values of

$$\frac{\sqrt{2} \pm \sqrt{1 + a}}{\pm \sqrt{1 - a}}.$$

17. Prove that $\sin \frac{\pi}{4} \operatorname{cosec} \frac{\pi}{12} + 4 \sin \frac{\pi}{10} = \sqrt{3} + \sqrt{5}.$

CHAPTER XVIII.

INVERSE CIRCULAR FUNCTIONS.

188. **Def.** $\sin^{-1} x$ stands for "The numerically smallest angle whose sine is x."

$\cos^{-1} x$ stands for "The numerically smallest angle whose cosine is x," etc.

Rule. When there are two numerically smallest angles take the positive one,

e.g. $$\cos^{-1}\tfrac{1}{2} = +60° \text{ and not } -60°.$$

N.B. $\sin^{-1}(\sin x) = \text{angle } x$; $\cos^{-1}(\cos x) = \text{angle } x$;

$$\tan^{-1}(\tan x) = \text{angle } x, \text{ etc.}$$

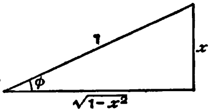
These equations put into words are seen to require no proof.

Some writers regard $\sin^{-1} x$ as many-valued; thus $\sin^{-1} x$ would equal $n\pi + (-1)^n \sin^{-1} x$; but in elementary work the student is advised to consider the above value only which is sometimes called The Principal Value.

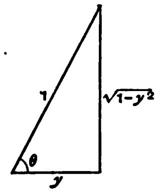
189. From a figure the student at once sees

$$\phi = \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

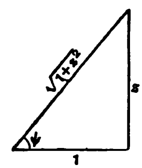
$$= \operatorname{cosec}^{-1}\frac{1}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}.$$



$$\begin{aligned}\theta = \cos^{-1} y &= \sin^{-1}\sqrt{1-y^2}\\ &= \tan^{-1}\frac{\sqrt{1-y^2}}{y}\\ &= \cot^{-1}\frac{y}{\sqrt{1-y^2}}\\ &= \sec^{-1}\frac{1}{y}\\ &= \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-y^2}},\end{aligned}$$



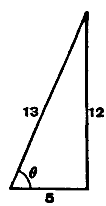
$$\begin{aligned}\psi = \tan^{-1} z &= \sin^{-1}\frac{z}{\sqrt{1+z^2}}\\ &= \cos^{-1}\frac{1}{\sqrt{1+z^2}}\\ &= \cot^{-1}\frac{1}{z}\\ &= \sec^{-1}\sqrt{1+z^2}\\ &= \operatorname{cosec}^{-1}\frac{\sqrt{1+z^2}}{z}.\end{aligned}$$



The above values need not be remembered, a figure at once recalls them.

190. A numerical example will make the above more clear. From the figure we see at once

$$\theta = \sin^{-1}\frac{12}{13} = \cos^{-1}\frac{5}{13} = \tan^{-1}\frac{12}{5} \text{ etc.}$$



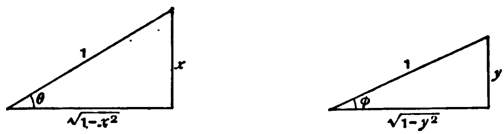
The addition and subtraction of inverse functions.

(i) *Find the value of*

$\sin^{-1} x \pm \sin^{-1} y$; *and of* $2\sin^{-1} x$.

191. Draw two figures putting

$$\sin^{-1} x = \theta; \quad \sin^{-1} y = \phi.$$



$$\sin^{-1} x + \sin^{-1} y = \theta + \phi$$

$$\begin{array}{l|l} = \sin^{-1}\{\sin(\theta+\phi)\} & = \cos^{-1}\{\cos(\theta+\phi)\}; \text{ etc.} \\ = \sin^{-1}\{\sin\theta\cos\phi + \sin\phi\cos\theta\} & = \cos^{-1}\{\cos\theta\cos\phi - \sin\theta\sin\phi\} \\ = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} & = \cos^{-1}\{\sqrt{1-x^2}\sqrt{1-y^2} - xy\}; \end{array}$$

obviously

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$$
$$= \cos^{-1}\{\sqrt{1-x^2}\sqrt{1-y^2} + xy\}.$$

192. $\sin^{-1} x \pm \sin^{-1} y = \theta \pm \phi = \tan^{-1}\{\tan(\theta \pm \phi)\}$

$$= \tan^{-1}\left\{\frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}\right\}$$

$$= \tan^{-1}\left\{\frac{\dfrac{x}{\sqrt{1-x^2}} \pm \dfrac{y}{\sqrt{1-y^2}}}{1 \mp \dfrac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}}\right\}.$$

193. $2\sin^{-1} x = 2\theta = \sin^{-1}(\sin 2\theta) = \cos^{-1}(\cos 2\theta)$; etc.

$$\begin{array}{l|l} = \sin^{-1}(2\sin\theta\cos\theta) & = \cos^{-1}(1 - 2\sin^2\theta) \\ = \sin^{-1}(2x\sqrt{1-x^2}) & = \cos^{-1}(1 - 2x^2). \end{array}$$

We leave it for the student to show

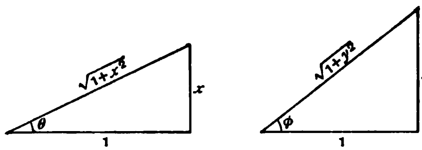
$$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\}$$
$$= \sin^{-1}\{y\sqrt{1-x^2} \pm x\sqrt{1-y^2}\} \text{ etc.}$$

and $\quad 2\cos^{-1} x = \cos^{-1}(2x^2 - 1) = \sin^{-1}(2x\sqrt{1-x^2}).$

194. (ii) *Find the value of*

$$\tan^{-1}x \pm \tan^{-1}y; \quad \textit{and of} \quad 2\tan^{-1}x.$$

Draw two figures putting $\tan^{-1}x = \theta$; $\tan^{-1}y = \phi$.



$$\tan^{-1}x + \tan^{-1}y = \theta + \phi$$

$$=\tan^{-1}\{\tan(\theta+\phi)\} \qquad = \sin^{-1}\{\sin(\theta+\phi)\};\ \text{etc.}$$

$$=\tan^{-1}\left\{\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}\right\} \qquad = \sin^{-1}\{\sin\theta\cos\phi+\cos\theta\sin\phi\}$$

$$=\tan^{-1}\frac{x+y}{1-xy} \qquad = \sin^{-1}\left\{\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+y^2}}+\frac{y}{\sqrt{1+y^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right\};$$

obviously $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$.

Again,

$$2\tan^{-1}x = 2\theta = \tan^{-1}(\tan 2\theta) \qquad = \cos^{-1}(\cos 2\theta);\ \text{etc.}$$

$$= \tan^{-1}\left\{\frac{2\tan\theta}{1-\tan^2\theta}\right\} \qquad = \cos^{-1}(2\cos^2\theta - 1)$$

$$= \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad = \cos^{-1}\left(\frac{2}{1+x^2}-1\right)$$

$$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

The values of $2\sin^{-1}x$, $2\cos^{-1}x$, $2\tan^{-1}x$ might obviously have been obtained from those of $\sin^{-1}x+\sin^{-1}y$, etc., by putting $x = y$.

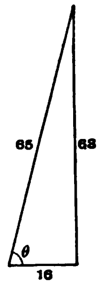
NUMERICAL EXAMPLES.

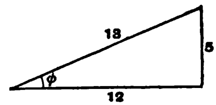
195. Ex. 1. Prove that

$$\cos^{-1}\tfrac{16}{65} - \cos^{-1}\tfrac{12}{13} = \sin^{-1}\tfrac{4}{5}.$$

Draw two figures putting

$$\cos^{-1}\tfrac{16}{65}=\theta;\quad \cos^{-1}\tfrac{12}{13}=\phi.$$





$$\begin{aligned}\cos^{-1}\tfrac{16}{65}-\cos^{-1}\tfrac{12}{13}&=\theta-\phi\\ &=\sin^{-1}\{\sin(\theta-\phi)\}\\ &=\sin^{-1}\{\sin\theta\cos\phi-\sin\phi\cos\theta\}\\ &=\sin^{-1}\left\{\frac{63}{65}\cdot\frac{12}{13}-\frac{5}{13}\cdot\frac{16}{65}\right\}\\ &=\sin^{-1}\left(\frac{4\,.\,169}{65\,.\,13}\right)=\sin^{-1}\tfrac{4}{5}.\end{aligned}$$

This example should be verified from Tables; thus

$$\begin{aligned}\cos^{-1}\tfrac{16}{65}&=\cos^{-1}(\cdot 2462)=75^\circ\,45',\\ \cos^{-1}\tfrac{12}{13}&=\cos^{-1}(\cdot 9231)=22^\circ\,37',\\ \sin^{-1}\tfrac{4}{5}&=\sin^{-1}(\cdot 8)\quad\;\;=53^\circ\,8',\end{aligned}$$

and $$75^\circ\,45'-22^\circ\,37'=53^\circ\,8'.$$

Ex. 2. Prove

$$\tan^{-1}\tfrac{2}{11}+2\tan^{-1}\tfrac{1}{7}=\tan^{-1}\tfrac{1}{2}.$$

Call $$\tan^{-1}\tfrac{2}{11}=\theta;\quad \tan^{-1}\tfrac{1}{7}=\phi;$$

then $$\begin{aligned}\tan^{-1}\tfrac{2}{11}+2\tan^{-1}\tfrac{1}{7}&=\theta+2\phi=\tan^{-1}\{\tan(\theta+2\phi)\}\\ &=\tan^{-1}\left\{\frac{\tan\theta+\tan 2\phi}{1-\tan\theta\tan 2\phi}\right\}\\ &=\tan^{-1}\left\{\frac{\tan\theta+\dfrac{2\tan\phi}{1-\tan^2\phi}}{1-\tan\theta\,.\,\dfrac{2\tan\phi}{1-\tan^2\phi}}\right\}\\ &=\tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}}\right\}\\ &=\tan^{-1}\left(\frac{48+77}{264-14}\right)=\tan^{-1}\tfrac{1}{2}.\end{aligned}$$

This should be verified from Tables, thus

$$\begin{aligned}\tan^{-1}\tfrac{2}{11}&=\;\;\tan^{-1}(\cdot 1818)=10^\circ\,18',\\ 2\tan^{-1}\tfrac{1}{7}&=2\tan^{-1}(\cdot 1429)=2\,(8^\circ\,8')=16^\circ\,16',\\ \tan^{-1}\tfrac{1}{2}&=\;\;\tan^{-1}(\cdot 5)\quad\;\;=26^\circ\,34',\end{aligned}$$

and $$10^\circ\,18'+16^\circ\,16'=26^\circ\,34'.$$

EXAMPLES XLI.

Complete the following:

1. $\sin^{-1}\frac{12}{13} = \tan^{-1}[\qquad]$.
2. $\cos^{-1}\frac{3}{5} = \cot^{-1}[\qquad]$.
3. $\tan^{-1}\frac{63}{16} = \sin^{-1}[\qquad]$.
4. $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}[\qquad]$.

Prove that

5. $\cos^{-1}\frac{63}{65} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{316}{845}$.
6. $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{63}{16}$.
7. $2\cos^{-1}\frac{3}{5} = \cos^{-1}(-\frac{7}{25})$.
8. $2\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{24}{25}$.
9. $2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{12}{5}$.
10. $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{17}{19}$.
11. $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{32}{43}$.
12. $\tan^{-1}\frac{3}{8} + \tan^{-1}\frac{2}{9} = \tan^{-1}\frac{43}{66}$.
13. $\tan^{-1}\frac{5}{2} + \tan^{-1}\frac{3}{4} = \tan^{-1}(-\frac{26}{7})$.
14. $\tan^{-1}\frac{7}{4} - \tan^{-1}\frac{1}{6} = \tan^{-1}\frac{38}{31}$.
15. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$.
16. $4\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{239} = 45°$.
17. $2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \frac{\pi}{4} - \tan^{-1}\frac{6}{61}$.
18. $\tan^{-1}n + \cot^{-1}(n+1) = \tan^{-1}(n^2+n+1)$.
19. $\tan^{-1}\left(\frac{x\sin a}{1-x\cos a}\right) - \tan^{-1}\left(\frac{x-\cos a}{\sin a}\right) = \frac{\pi}{2} - a$.
20. $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$.
21. $\cos^{-1}\frac{a-x}{a+x} = 2\tan^{-1}\sqrt{\frac{x}{a}}$.

22. $\sin^{-1}\left(\dfrac{x-a+b}{2b}\right)^{\frac{1}{2}} = \frac{1}{2}\cos^{-1}\left(\dfrac{a-x}{b}\right)$.

23. $\cot^{-1}\dfrac{ab+1}{a-b} + \cot^{-1}\dfrac{bc+1}{b-c} + \cot^{-1}\dfrac{ca+1}{c-a} = 0$.

24. $2\cos^{-1}\sqrt{\dfrac{a-x}{a-b}} = \sin^{-1}\dfrac{2\sqrt{(a-x)(x-b)}}{a-b}$.

25. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \dfrac{\pi}{4}$.

26. $\tan^{-1}x + \tan^{-1}\dfrac{1-x}{1+x} = \tan^{-1}\left(\dfrac{1+x^2}{1+2x-x^2}\right)$.

27. $\sin^{-1}\dfrac{\sqrt{5}}{5} + \cot^{-1}3 = \dfrac{\pi}{4}$.

28. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-yz-zx-xy}\right)$.

29. $x^2 = \sin 2y$, when

$$y = \tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}.$$

Solution of equations.

196. This is best illustrated by actual examples.

(i) Solve

$$\tan^{-1}x + \tan^{-1}(1-x) = 2\tan^{-1}\sqrt{x-x^2}.$$

From Art. 194

$$\tan^{-1}\frac{x+(1-x)}{1-x(1-x)} = \tan^{-1}\frac{2\sqrt{x-x^2}}{1-(x-x^2)},$$

$$\therefore \quad \frac{1}{1-x+x^2} = \frac{2\sqrt{x-x^2}}{1-x+x^2},$$

therefore either

$$1-x+x^2 = 0, \quad i.e. \quad x = \frac{1 \pm \sqrt{-3}}{2},$$

or

$$2\sqrt{x-x^2} = 1, \quad i.e. \quad x = \tfrac{1}{2}.$$

$$Ans. \quad \tfrac{1}{2} \text{ or } \frac{1 \pm \sqrt{-3}}{2}.$$

(ii) Solve

$$\sin^{-1} 2x = \sin^{-1} x\sqrt{3} + \sin^{-1} x.$$

From Art. 191,

$$\sin^{-1} 2x = \sin^{-1}\{x\sqrt{3}\sqrt{1-x^2} + x\sqrt{1-3x^2}\},$$

$$\therefore\ 2x = x\sqrt{3}\sqrt{1-x^2} + x\sqrt{1-3x^2},$$

therefore either $x = 0,$

or $2 = \sqrt{3}\sqrt{1-x^2} + \sqrt{1-3x^2},$

i.e. $4 + (1-3x^2) - 4\sqrt{1-3x^2} = 3(1-x^2),$

$$\therefore\ 4\sqrt{1-3x^2} = 2,$$

$$1 - 3x^2 = \tfrac{1}{4},$$

$$\therefore\ x^2 = \tfrac{1}{4},\quad \therefore\ x = \pm\tfrac{1}{2}.$$

Ans. $0;\ \pm\frac{1}{2}$.

EXAMPLES XLII.

Solve

1. $\tan^{-1} 2x + \tan^{-1} 3x = \dfrac{\pi}{4}.$

2. $\tan^{-1}\dfrac{1}{a-1} = \tan^{-1}\dfrac{1}{x} + \tan^{-1}\dfrac{1}{a^2-x+1}.$

3. $\tan^{-1}\dfrac{x+1}{x-1} + \tan^{-1}\dfrac{x-1}{x} = \tan^{-1}(-9).$

4. $\tan^{-1}(ax+b) + \tan^{-1}(ax-b) = \dfrac{\pi}{4}.$

5. $\sin^{-1} x + \sin^{-1}\dfrac{x}{2} = \dfrac{\pi}{4}.$

6. $\operatorname{cosec}^{-1} x = \operatorname{cosec}^{-1} a + \operatorname{cosec}^{-1} b.$

7. $\cos^{-1}\dfrac{1}{\sqrt{1+x^2}} - \cos^{-1}\dfrac{x}{\sqrt{1+x^2}} = \sin^{-1}\dfrac{1+x}{1+x^2}.$

8. $\sin^{-1}\dfrac{5}{x}+\sin^{-1}\dfrac{12}{x}=\dfrac{\pi}{2}$.

9. Solve $$\tan^{-1}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=30^\circ.$$

10. $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$.

11. $\sin^{-1}x+\sin^{-1}\frac{1}{2}=\sin^{-1}\frac{3}{4}$.

12. $\sin^{-1}\dfrac{12}{13}+\sin^{-1}\dfrac{12}{x}=\dfrac{\pi}{2}$.

13. $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$.

14. $\cot^{-1}x-\cot^{-1}(x+2)=15^\circ$.

15. $\sec^{-1}\dfrac{x}{a}-\sec^{-1}\dfrac{x}{b}=\sec^{-1}b-\sec^{-1}a$.

16. $\cot^{-1}(x-a)+\cot^{-1}(x-b)+\cot^{-1}(x-c)=0$.

17. If $\tan^{-1}a=\operatorname{cosec}^{-1}a=\cos^{-1}b$, prove that one value of b is $\dfrac{\sqrt{5}-1}{2}$.

CHAPTER XIX.

ELIMINATION.

197. FROM certain equations it is frequently desirable to deduce others which shall not contain certain variables. This process is called Elimination and the result obtained the **Eliminant**.

If the number of equations given is one greater than the number of variables, it is always possible to eliminate those variables.

Each problem must be considered on its own merits.

Ex. 1. Eliminate θ between

$$a\cos\theta + b\sin\theta = c,$$

and

$$a'\cos\theta + b'\sin\theta = c'.$$

Solving for $\cos\theta$ and $\sin\theta$, we obtain

$$\cos\theta = \frac{bc' - b'c}{ba' - b'a},$$

$$\sin\theta = \frac{a'c - ac'}{ba' - b'a}.$$

$\therefore$ squaring and adding,

$$1 = \left(\frac{bc' - b'c}{ba' - b'a}\right)^2 + \left(\frac{a'c - ac'}{ba' - b'a}\right)^2,$$

or

$$(ba' - b'a)^2 = (bc' - b'c)^2 + (a'c - ac')^2.$$

Ex. 2. Eliminate θ between

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \text{.....................(i),}$$

and

$$\tan\theta = c \quad \text{..........................(ii).}$$

From (ii)

$$\frac{\sin\theta}{c} = \frac{\cos\theta}{1} = \sqrt{\frac{\sin^2\theta + \cos^2\theta}{c^2+1}}$$

$$= \frac{1}{\sqrt{c^2+1}},$$

$\therefore$ substituting in (i)

$$a\sqrt{c^2+1}\,x - \frac{b\sqrt{c^2+1}}{c}\,y = a^2 - b^2.$$

Ex. 3. Eliminate θ between

$$\frac{\sec^4\phi - 1}{\sec^4\phi + 1} = \frac{x}{a} \quad \text{..........................(i),}$$

and

$$\sec^2\phi + \cos^2\phi = \frac{2b}{y} \quad \text{..........................(ii).}$$

From (ii)

$$\frac{b}{y} = \frac{\sec^4\phi + 1}{2\sec^2\phi},$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(\sec^4\phi - 1)^2 + 4\sec^4\phi}{(\sec^4\phi + 1)^2}$$

$$= \frac{(\sec^4\phi + 1)^2}{(\sec^4\phi + 1)^2} = 1.$$

Ex. 4. Eliminate θ and ϕ between

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{.....................(i),}$$

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1 \quad \text{.....................(ii),}$$

$$\theta - \phi = 2\alpha \quad \text{.................(iii).}$$

Solving for $\frac{x}{a}$ and $\frac{y}{b}$ from (i) and (ii)

$$\frac{\frac{x}{a}}{\sin\theta - \sin\phi} = \frac{\frac{y}{b}}{\cos\phi - \cos\theta} = \frac{1}{\sin(\theta - \phi)} = \frac{1}{\sin 2\alpha},$$

$$\therefore \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{(\sin\theta - \sin\phi)^2 + (\cos\phi - \cos\theta)^2}{\sin^2 2a}$$
$$= \frac{2 - 2(\sin\theta\sin\phi + \cos\theta\cos\phi)}{\sin^2 2a}$$
$$= \frac{2 - 2\cos 2a}{\sin^2 2a}$$
$$= \frac{4\sin^2 a}{4\sin^2 a\cos^2 a} = \frac{1}{\cos^2 a}.$$

Ex. 5. Eliminate θ and ϕ between

$$x\cos\theta + y\sin\theta = x\cos\phi + y\sin\phi = 1,$$

and
$$a\cos\theta\cos\phi + b\sin\theta\sin\phi + c + g(\cos\theta + \cos\phi) + f(\sin\theta + \sin\phi) + h\sin(\theta+\phi) = 0.$$

From
$$x\cos\theta + y\sin\theta = 1,$$
$$x\cos\phi + y\sin\phi = 1,$$

we obtain
$$\frac{\cos\frac{1}{2}(\theta+\phi)}{x} = \frac{\sin\frac{1}{2}(\theta+\phi)}{y} = \frac{\cos\frac{1}{2}(\theta-\phi)}{1} \quad \text{......(i)},$$

$$\therefore \text{ Each fraction} = \frac{\sin\frac{1}{2}(\theta-\phi)}{\sqrt{x^2+y^2-1}} \text{ and also} = \frac{1}{\sqrt{x^2+y^2}} \quad \text{.........(ii)}.$$

The third equation may be written

$$a[\cos(\theta+\phi) + \cos(\theta-\phi)] + b[\cos(\theta-\phi) - \cos(\theta+\phi)]$$
$$+ 2c + 4g[\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)] + 4f[\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)]$$
$$+ 4h\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta+\phi) = 0 \quad \text{..............................(iii)}.$$

From (i) and (ii),

$$\cos(\theta+\phi) = 2\cos^2\tfrac{1}{2}(\theta+\phi) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2},$$
$$\cos(\theta-\phi) = 2\cos^2\tfrac{1}{2}(\theta-\phi) - 1 = \frac{2}{x^2+y^2} - 1 = \frac{2-x^2-y^2}{x^2+y^2}.$$

Therefore, substituting in (iii)

$$a\frac{x^2-y^2+2-x^2-y^2}{x^2+y^2} + b\frac{2-x^2-y^2-x^2+y^2}{x^2+y^2} + 2c + 4g\frac{x}{x^2+y^2}$$
$$+ 4f\frac{y}{x^2+y^2} + 4h\frac{xy}{x^2+y^2} = 0,$$

$$a(1-y^2) + b(1-x^2) + c(x^2+y^2) + 2gx + 2fy + 2hxy = 0.$$

EXAMPLES XLIII.

Eliminate ϕ between the equations:

1. $x = a\cos^2\phi, \quad y = b\sin^2\phi.$

2. $x = \sin\phi - \operatorname{cosec}\phi, \quad y = \cos\phi - \sec\phi.$

3. $\dfrac{\cos\phi}{h} = \dfrac{\sin\phi}{k} = \dfrac{c\cos\phi + b}{a^2}.$

4. $\sin\theta = a\cos\phi + b\sin\phi, \quad \cos\theta = a\sin\phi - b\cos\phi.$

5. $x = \sin\phi + \cos\phi, \quad y = \tan\phi + \cot\phi.$

6. $x = \sin\phi + \tan\phi, \quad y = \sin\phi - \tan\phi.$

7. $x\sin\phi - y\cos\phi = \sqrt{x^2+y^2}, \quad \dfrac{\sin^2\phi}{a^2} + \dfrac{\cos^2\phi}{b^2} = \dfrac{1}{x^2+y^2}.$

8. $x = a\cot^2\phi, \quad y = 2a\tan\phi.$

9. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1, \quad -\dfrac{x}{a}\sin\phi + \dfrac{y}{b}\cos\phi = 1.$

10. $x = 3\cos\phi + \cos 3\phi, \quad y = 3\sin\phi - \sin 3\phi.$

11. $x = a\cos\phi\,(4\cos^2\phi - 3), \quad y = b\sin\phi\,(4\cos^2\phi - 1).$

12. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1, \quad x\sin\phi - y\cos\phi$
$= (a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}}.$

Eliminate θ and ϕ from:

13. $a\sin\theta - b\sin\phi = 0, \quad c\cos\theta - d\cos\phi = 0, \quad \theta - 2\phi = 0.$

14. $\tan\theta + \tan\phi = a, \quad \cot\theta + \cot\phi = b, \quad \theta + \phi = \alpha.$

15. $\tan\theta + \tan\phi = a, \quad \cot\theta + \cot\phi = b, \quad \theta - \phi = \alpha.$

16. $\cos\theta + \cos\phi = a, \quad \cot\theta + \cot\phi = b, \quad \operatorname{cosec}\theta + \operatorname{cosec}\phi = c.$

17. $\sin\alpha\cos\theta = \sin\beta, \quad \sin\alpha\cos\phi = \sin\gamma, \quad \cos(\theta - \phi)$
$= \sin\beta\sin\gamma.$

18. $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2 = \dfrac{ax}{\cos\phi} - \dfrac{by}{\sin\phi}$, $\theta - \phi = \dfrac{\pi}{2}$.

19. $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$, $\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2} = c$.

Eliminate θ between:

20. $a\sin\theta + b\tan\theta = m$, $a\cos\theta + b\cot\theta = n$.

21. $x\cos\theta + y\sin\theta = a\cos 2\theta$, $x\sin\theta - y\cos\theta = 2a\sin 2\theta$.

22. $\operatorname{cosec}\theta - \sin\theta = m$, $\sec\theta - \cos\theta = n$.

23. $x\cos(\theta + a) + y\sin(\theta + a) = a\sin 2\theta$,
$y\cos(\theta + a) - x\sin(\theta + a) = 2a\cos 2\theta$.

24. $(a + b)\tan(\theta - \phi) = (a - b)\tan(\theta + \phi)$,
$a\cos 2\phi + b\cos 2\theta = c$.

25. $x = 2a\cos\theta + a\cos 2\theta$,
$y = 2a\sin\theta - a\sin 2\theta$.

CHAPTER XX.

INEQUALITIES AND LIMITS.

Throughout this chapter

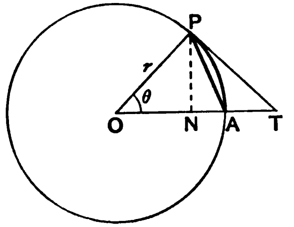
$\theta =$ the *Circular Measure* of a positive *Acute* angle.

198. To show $\quad \tan\theta > \theta > \sin\theta.$

Draw a circle radius r, centre O, and let

$$A\hat{O}P = \theta.$$

Draw PT a tangent and PN perpendicular to OA.



Then from fig.

$$\triangle\ OPT > \text{sector } AOP > \triangle\ AOP,$$

$$\therefore\ \tfrac{1}{2}OP\,.\,PT > \tfrac{1}{2}r^2\theta$$

$$> \tfrac{1}{2}PN\,.\,OA,$$

$$\therefore\ \tfrac{1}{2}r\,.\,r\tan\theta > \tfrac{1}{2}r^2\,\theta > \tfrac{1}{2}r\sin\theta\,.\,r,$$

$$i.e.\quad \tan\theta > \theta > \sin\theta.$$

199. The limiting values of

$$\frac{\sin\theta}{\theta} \text{ and } \frac{\tan\theta}{\theta},$$

when θ is indefinitely diminished, are each **unity**.

By previous art.

$$\tan\theta > \theta > \sin\theta;$$

$$\therefore \quad \sec\theta > \frac{\theta}{\sin\theta} > 1.$$

But in the limit when θ is indefinitely diminished

$$\sec\theta = 1,$$

hence $$\operatorname*{Lt}_{\theta=0} \frac{\theta}{\sin\theta} = 1,$$

and therefore $$\operatorname*{Lt}_{\theta=0} \frac{\sin\theta}{\theta} = 1;$$

again $$\tan\theta > \theta > \sin\theta;$$

$$\therefore \quad 1 > \frac{\theta}{\tan\theta} > \cos\theta.$$

But in the limit when θ is indefinitely diminished

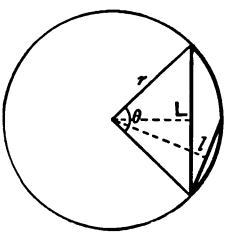
$$\cos\theta = 1,$$

hence $$\operatorname*{Lt}_{\theta=0} \frac{\theta}{\tan\theta} = 1,$$

and therefore $$\operatorname*{Lt}_{\theta=0} \frac{\tan\theta}{\theta} = 1.$$

Ex. The arc (if small) of a circle

$$= \frac{8l - L}{3} \text{ approx.}$$



where l = chord of half the arc,

L = chord of whole arc.

Suppose r = radius, θ angle subtended at centre by arc,

$$\frac{8l - L}{3} = \frac{8 \cdot 2r \sin\frac{\theta}{4} - 2r\sin\frac{\theta}{2}}{3}$$

$$= \frac{2r}{3}\left(8 \cdot \frac{\theta}{4} - \frac{\theta}{2}\right) \text{ when } \theta \text{ is small,}$$

$$= r\theta = \text{the arc.}$$

200. Limits in sexagesimal measure.

If x'' = the sexagesimal measure of the angle θ radians,

then $$\frac{x\pi}{180 \times 60 \times 60} = \theta,$$

and $$\frac{\sin x}{x} = \frac{\sin\theta}{\theta} \cdot \frac{\pi}{180 \times 60 \times 60},$$

and $$\frac{\tan x}{x} = \frac{\tan\theta}{\theta} \cdot \frac{\pi}{180 \times 60 \times 60}.$$

Hence

$$\operatorname*{Lt}_{x=0} \frac{\sin x}{x} = \operatorname{Lt} \frac{\tan x}{x} = \frac{\pi}{180 \times 60 \times 60}.$$

201. To show

$$\sin\theta > \theta - \frac{\theta^3}{4},$$

$$\cos\theta > 1 - \frac{\theta^2}{2}.$$

$$\begin{aligned} \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} \\ &= 2\tan\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right) \\ &= \theta \cdot \frac{\tan\frac{\theta}{2}}{\frac{\theta}{2}}\left\{1 - \frac{\theta^2}{4}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2\right\}. \end{aligned}$$

Now by Art. 198,

$$\frac{\tan\frac{\theta}{2}}{\frac{\theta}{2}} > 1 \text{ and } \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} < 1;$$

$$\therefore \sin\theta > \theta\left\{1 - \frac{\theta^2}{4}\right\} > \theta - \frac{\theta^3}{4};$$

again $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$= 1 - 2.\frac{\theta^2}{4}\left\{\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right\}^2$$

$$> 1 - \frac{\theta^2}{2}.$$

202. To show

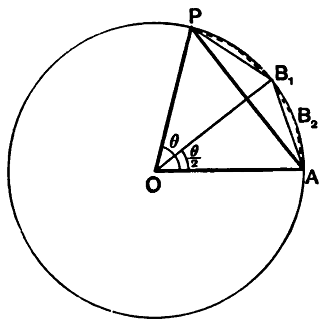
(i) $\quad \sin\theta > \theta - \dfrac{\theta^3}{6}$,

(ii) $\quad \cos\theta < 1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$,

(iii) $\quad \tan\theta > \theta + \dfrac{\theta^3}{3} + \dfrac{\theta^5}{8}$.

(i) Draw a circle radius r.

Let $\hat{AOP}$ = the angle θ,
OB_1 bisect AOP,
OB_2 bisect AOB_1,
etc.



Area of sector
$AOP = \triangle AOP + \triangle APB_1 + 2\triangle AB_1B_2 + 2^2\triangle AB_2B_3$ + etc. to infinity.

$$\therefore \tfrac{1}{2}r^2\theta = \tfrac{1}{2}r^2\sin\theta + \tfrac{1}{2}AB_1^2.\sin P\hat{B}_1A$$
$$+ 2.\tfrac{1}{2}AB_2^2.\sin A\hat{B}_2B_1 + 2^2.\tfrac{1}{2}AB_3^2.\sin A\hat{B}_3B_2 + \ldots\ldots$$

$$= \tfrac{1}{2}r^2\sin\theta + \tfrac{1}{2}\left(2r\sin\frac{\theta}{4}\right)^2.\sin\frac{\theta}{2}$$
$$+ 2.\tfrac{1}{2}\left(2r.\sin\frac{\theta}{8}\right)^2.\sin\frac{\theta}{4} + 2^2.\tfrac{1}{2}\left(2r\sin\frac{\theta}{16}\right)^2.\sin\frac{\theta}{8} + \ldots\ldots$$

$$< \tfrac{1}{2}r^2\sin\theta + \tfrac{1}{2}r^2\left(2.\frac{\theta}{4}\right)^2\frac{\theta}{2} + 2.\frac{r^2}{2}\left(2.\frac{\theta}{2^3}\right)^2.\frac{\theta}{2^2}$$
$$+ 2^2.\frac{r^2}{2}\left(2.\frac{\theta}{2^4}\right)^2.\frac{\theta}{2^3} + \ldots\ldots;$$

$$\therefore \quad \theta < \sin\theta + \frac{\theta^3}{8}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\ldots\right)$$

$$< \sin\theta + \frac{\theta^3}{8}\cdot\frac{1}{1-\frac{1}{4}}$$

$$< \sin\theta + \frac{\theta^3}{8}\cdot\frac{4}{3};$$

$$\therefore \quad \sin\theta > \theta - \frac{\theta^3}{6}.$$

(ii) $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$< 1 - 2\left\{\frac{\theta}{2} - \frac{\left(\frac{\theta}{2}\right)^3}{6}\right\}^2$$

$$< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{1152}.$$

Hence $$\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

Lemma (A) $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$ is positive,

$$\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

and $\cos\theta$ is positive since θ is a positive acute angle.

$$\therefore \; 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} \text{ is positive.}$$

Lemma (B) $\frac{7\theta^7}{144} - \frac{\theta^9}{192}$ is positive,

$$\frac{7\theta^7}{144} - \frac{\theta^9}{192} = \frac{\theta^7}{192}\left\{\frac{7\times 192}{144} - \theta^2\right\}$$

$$= \frac{\theta^7}{192}\cdot\frac{1}{4}\cdot\left\{37\tfrac{1}{3} - (2\theta)^2\right\}.$$

Now
$$(2\theta)^2 < \pi^2$$
$$< 16$$
$$< 37\tfrac{1}{3};$$
$$\therefore \frac{7\theta^7}{144} - \frac{\theta^9}{192} \text{ is positive.}$$

(iii)
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$> \frac{\theta - \dfrac{\theta^3}{6}}{1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}}$$
$$> \theta + \frac{\theta^3}{3} + \frac{\theta^5}{8} + \frac{\dfrac{7\theta^7}{144} - \dfrac{\theta^9}{192}}{1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}}, \text{ by division}$$
$$> \theta + \frac{\theta^3}{3} + \frac{\theta^5}{8}.$$

203. To show that $\dfrac{\sin\theta}{\theta}$ continually decreases as θ increases from 0 to $\dfrac{\pi}{2}$.

We have only to show
$$\frac{\sin\theta}{\theta} - \frac{\sin(\theta+\alpha)}{\theta+\alpha}$$
is positive when α is acute.

$$\text{Expression} = \frac{(\theta+\alpha)\sin\theta - \theta . \sin(\theta+\alpha)}{\theta(\theta+\alpha)}$$
$$= \frac{\theta\sin\theta(1-\cos\alpha) + (\alpha\sin\theta - \theta\cos\theta\sin\alpha)}{\theta(\theta+\alpha)}$$
$$= \text{a positive quantity} + \frac{\alpha\sin\theta - \theta\cos\theta\sin\alpha}{\theta(\theta+\alpha)}$$

$$= \text{a positive quantity} + \frac{\dfrac{\dfrac{\tan\theta}{\theta} - \dfrac{\sin\alpha}{\alpha}}{\theta(\theta+\alpha)}}{\theta\alpha\cos\theta}$$

$$= \text{a positive quantity},$$

because $$\frac{\tan\theta}{\theta} > 1 \quad \text{and} \quad \frac{\sin\alpha}{\alpha} < 1.$$

In a similar way $\dfrac{\tan\theta}{\theta}$ continually increases.

204. Euler's Theorem.

$$\operatorname*{Lt}_{n=\infty} \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n} \text{ is } \frac{\sin\theta}{\theta}.$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}$$

$$= 2^3\sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}, \text{ etc.}$$

$$\therefore \frac{\sin\theta}{2^n\sin\dfrac{\theta}{2^n}} = \cos\frac{\theta}{2^n}\cos\frac{\theta}{2^{n-1}}\ldots\ldots\cos\frac{\theta}{2^2}\cos\frac{\theta}{2};$$

thus when n is indefinitely increased

$$\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n} = \operatorname*{Lt}_{n=\infty}\frac{\sin\theta}{\theta\dfrac{\sin\dfrac{\theta}{2^n}}{\dfrac{\theta}{2^n}}}$$

$$= \frac{\sin\theta}{\theta}.$$

205. Ex. 1. Find the values of $\sin 8'$ and $\cos 8'$ $[\pi = 3{\cdot}14159]$.

$$8' = \frac{8\pi}{180 \times 60} \text{ radians} = \frac{8 \times 3{\cdot}14159}{180 \times 60} \text{ (approx.)}$$
$$= {\cdot}0023271 \text{ radians};$$

therefore, since $$\sin\theta = \theta - \frac{\theta^3}{4} \text{ (approx.)},$$

$$\sin 8' = {\cdot}0023271 - \frac{({\cdot}0023271)^3}{4}$$
$$= {\cdot}0023271 - {\cdot}000000002\ldots$$
$$= {\cdot}0023271 \text{ (nearly)}.$$

Also $$\cos\theta = 1 - \frac{\theta^2}{2} \text{ (approx.)}$$
$$= 1 - \tfrac{1}{2}({\cdot}0023271)^2$$
$$= 1 - {\cdot}0000027$$
$$= {\cdot}9999973.$$

Ex. 2. If $\dfrac{\sin\theta}{\theta} = \dfrac{483}{484}$ find an approximate value for θ.

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{4}}{\theta} = 1 - \frac{\theta^2}{4};$$
$$\therefore\ 1 - \frac{\theta^2}{4} = \frac{483}{484}$$

or $$\theta^2 = \tfrac{1}{121};$$
$$\therefore\ \theta = \tfrac{1}{11} \text{ radian}.$$

Ex. 3. Solve approximately $\sin\left(\dfrac{\pi}{3} + \theta\right) = {\cdot}87$.

Expanding $$\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = {\cdot}87,$$

and since
$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = {\cdot}866,$$

θ is a small angle,
$$\therefore\ \frac{\sqrt{3}}{2} + \frac{1}{2}\theta = {\cdot}87;$$
$$\therefore\ \theta = 1{\cdot}74 - 1{\cdot}73205$$
$$= {\cdot}00795 \text{ radian}.$$

Ex. 4. Find the angle subtended by a kilometre-stone 1 metre high at a place 1·5 kilometres off $[\pi = \frac{22}{7}]$.

If θ is the number of radians in the angle

$$\theta = \tan\theta = \frac{1 \text{ metre}}{1{\cdot}5 \text{ kilom.}}$$
$$= \tfrac{1}{1500} \text{ radians}$$
$$= \frac{180 \times 7 \times 60'}{22 \times 1500} = \frac{126'}{55}$$
$$= 2'{\cdot}29.$$

EXAMPLES XLIV.

1. Find the values of $\sin 5'$ and $\cos 5'$ $[\pi = 3{\cdot}14159]$.

2. If $\frac{\sin\theta}{\theta} = \frac{675}{676}$, find the value of θ.

3. Find the values of $\sin 3'$ and $\cos 3'$ $[\pi = 3{\cdot}14159]$.

4. Solve the equation $\sin\left(\frac{\pi}{4} + \theta\right) = {\cdot}71$.

5. Calculate the approximate value of θ, when $\cos\theta = \frac{1681}{1682}$.

6. If $\sin\theta = \frac{1155}{1156}\theta$, find the value of θ.

7. Find the value of θ from the equation $\cos\left(\frac{\pi}{3} - \theta\right) = {\cdot}51$.

8. Find θ when $\cos\theta = \frac{2737}{2738}$.

9. Solve the equation $\tan\left(\frac{\pi}{3} + \theta\right) = 1{\cdot}73$.

10. If $\frac{\sin\theta}{\theta} = \frac{1763}{1764}$, calculate the approximate value of θ.

11. A post, 1 foot high, stands at the top of a tower of height 150 feet; calculate (to $\frac{1}{100}$ of a minute) the angle it subtends at a point on the ground 900 feet from the foot of the tower. $[\pi = \frac{22}{7}.]$

12. Find the value of $\cos\left(\frac{\pi}{6} + \theta\right)$, when $\theta = {\cdot}005$ radian.

13. An object, 880 metres off, subtends an angle of 51′ at the observer's eye: find the length of the object. $(\pi = \frac{22}{7}.)$ [Answer to 1 centimetre.]

14. A cliff 180 metres high is surmounted by a flagstaff which subtends an angle of ·035 radian at a point on the ground 350 metres from the foot of the cliff. Find the height of the flagstaff to the nearest decimetre.

CHAPTER XXI.

SUMMATION OF SERIES.

206. To find the sum of the sines of a series of angles in A.P.

Let $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$
$+ \sin\{\alpha + (n-1)\beta\} = S;$

$$\therefore\ 2S\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + \ldots\ldots + 2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}.$$

Now
$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right)$$

....

$$2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{2n-3}{2}\beta\right) - \cos\left(\alpha + \frac{2n-1}{2}\beta\right);$$

therefore, by addition

$$2S\sin\frac{\beta}{2}=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$=2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2};$$

$$\therefore\ S=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

207. To find the sum of the cosines of a series of angles in A.P.

Let $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots$
$+\cos\{\alpha+(n-1)\beta\}=C.$

The value of this series may either be deduced from the last by putting $\alpha=\frac{\pi}{2}+\alpha$, whence we obtain

$$C=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}},$$

or may be worked out independently.

$$2C\sin\frac{\beta}{2}=2\cos\alpha\sin\frac{\beta}{2}+2\cos(\alpha+\beta)\sin\frac{\beta}{2}+\ldots\ldots$$
$$+2\cos\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}.$$

Now $$2\cos\alpha\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$

$$2\cos(\alpha+\beta)\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)$$

$$2\cos(\alpha+2\beta)\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{5\beta}{2}\right)-\sin\left(\alpha+\frac{3\beta}{2}\right)$$

....

$$2\cos\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)$$
$$-\sin\left(\alpha+\frac{2n-3}{2}\beta\right);$$

therefore, by addition

$$2\mathrm{C}\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$
$$=2\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2};$$

$$\therefore\ \mathrm{C}=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

208. It should be observed that

$$\alpha+\frac{n-1}{2}\beta=\tfrac{1}{2}[\alpha+\{\alpha+(n-1)\beta\}]$$
$$=\tfrac{1}{2}\,(\text{sum of first and last angle}),$$

and that the two results only differ in the first term of the numerator.

209. $\sin\frac{n\beta}{2}=0$, when $\frac{n\beta}{2}=k\pi$ or $\beta=\frac{2k\pi}{n}$,

k being an integer; and in this case both S and C vanish.

Thus the sum of the sines or cosines of n angles in A.P. vanishes when the common difference of the angle, β, is a multiple of $\frac{2\pi}{n}$.

210. If in the sine-series, we change β into $\beta+\pi$, we obtain

$$\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{\sin\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\cos\frac{\beta}{2}}$$

and in the cosine-series

$$\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{\cos\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\cos\frac{\beta}{2}}.$$

211. Ex. 1. Find the value of $\sin A+\sin 3A+\sin 5A+\ldots$ to n terms.

By Art. 206,

$$\text{Series}=\frac{\sin\left(A+\frac{n-1}{2}2A\right)\sin n\frac{2A}{2}}{\sin\frac{2A}{2}}$$

$$=\frac{\sin^2 nA}{\sin A}.$$

Ex. 2. Find the value of $\cos\frac{\pi}{17}+\cos\frac{3\pi}{17}+\ldots+\cos\frac{15\pi}{17}$.

By Art. 207,

$$\text{Series}=\frac{\cos\left(\frac{\pi}{17}+\frac{7}{2}\cdot\frac{2\pi}{17}\right)\sin\frac{8\pi}{17}}{\sin\frac{\pi}{17}}$$

$$=\frac{\cos\frac{8\pi}{17}\sin\frac{8\pi}{17}}{\sin\frac{\pi}{17}}=\frac{1}{2}\,\frac{\sin\frac{16\pi}{17}}{\sin\frac{\pi}{17}}=\frac{1}{2}.$$

212. Several other series can be summed by decomposing each term into the difference of two others.

Ex. 3. Find the value of

$$\sin\alpha\cos 5\alpha + \sin 3\alpha\cos 7\alpha + \sin 5\alpha\cos 9\alpha + \dots \text{ to } n \text{ terms.}$$

$$\begin{aligned} 2S &= (\sin 6\alpha - \sin 4\alpha) + (\sin 10\alpha - \sin 4\alpha) + (\sin 14\alpha - \sin 4\alpha) + \dots \\ &= (\sin 6\alpha + \sin 10\alpha + \sin 14\alpha + \dots) \\ &\quad - (\sin 4\alpha + \sin 4\alpha + \sin 4\alpha + \dots) \\ &= \frac{\sin(2n\alpha + 4\alpha)\sin 2n\alpha}{\sin 2\alpha} - n\sin 4\alpha. \end{aligned}$$

Ex. 4. Find the value of

$$\frac{1}{\sin\alpha\sin 3\alpha} + \frac{1}{\sin 3\alpha\sin 5\alpha} + \frac{1}{\sin 5\alpha\sin 7\alpha} + \dots.$$

Since $\dfrac{\sin 2\alpha}{\sin\alpha\sin 3\alpha} = \dfrac{\sin(3\alpha - \alpha)}{\sin\alpha\sin 3\alpha} = \cot\alpha - \cot 3\alpha$

$$\frac{\sin 2\alpha}{\sin 3\alpha\sin 5\alpha} = \frac{\sin(5\alpha - 3\alpha)}{\sin 3\alpha\sin 5\alpha} = \cot 3\alpha - \cot 5\alpha$$

....

$$\begin{aligned} \frac{\sin 2\alpha}{\sin(2n-1)\alpha\sin(2n+1)\alpha} &= \frac{\sin(\overline{2n+1}\,\alpha - \overline{2n-1}\,\alpha)}{\sin(2n-1)\alpha\sin(2n+1)\alpha} \\ &= \cot(2n-1)\alpha - \cot(2n+1)\alpha; \end{aligned}$$

therefore, adding

$$\sin 2\alpha \,.\, S = \cot\alpha - \cot(2n+1)\alpha$$

or $$S = \frac{\cot\alpha - \cot(2n+1)\alpha}{\sin 2\alpha}.$$

Ex. 5. Find the value of

$$\operatorname{cosec}\alpha + \operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha + \dots \text{ to } n \text{ terms.}$$

Since $\operatorname{cosec}\alpha = \cot\dfrac{\alpha}{2} - \cot\alpha$

$$\operatorname{cosec} 2\alpha = \cot\alpha - \cot 2\alpha$$

....

$$\operatorname{cosec} 2n\alpha = \cot n\alpha - \cot 2n\alpha;$$

therefore, adding $S = \cot\dfrac{\alpha}{2} - \cot 2n\alpha.$

Ex. 6. Find the value of

$$\tan a + \frac{1}{2}\tan\frac{a}{2} + \frac{1}{2^2}\tan\frac{a}{2^2} + \ldots \text{ to } n \text{ terms.}$$

$$\tan a = \cot a - 2\cot 2a$$

$$\frac{1}{2}\tan\frac{a}{2} = \frac{1}{2}\cot\frac{a}{2} - \cot a$$

$$\frac{1}{2^2}\tan\frac{a}{2^2} = \frac{1}{2^2}\cot\frac{a}{2^2} - \frac{1}{2}\cot\frac{a}{2}$$

....

$$\frac{1}{2^{n-1}}\tan\frac{a}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{a}{2^{n-2}};$$

therefore, adding $$S = \frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}} - 2\cot 2a.$$

Series involving the squares and cubes of sines and cosines may be evaluated by transforming them into new series containing multiple angles.

Ex. 7. Find the value of

$$\cos^2 a + \cos^2(a+\beta) + \cos^2(a+2\beta) + \ldots \text{ to } n \text{ terms.}$$

Since $$2\cos^2 a = 1 + \cos 2a,$$

$$\begin{aligned} \therefore\ 2S &= \{1+\cos 2a\} + \{1+\cos 2(a+\beta)\} + \{1+\cos 2(a+2\beta)\} + \ldots \\ &= n + \cos 2a + \cos(2a+2\beta) + \cos(2a+4\beta) + \ldots \\ &= n + \frac{\cos\{2a+(n-1)\beta\}\sin n\beta}{\sin\beta}. \end{aligned}$$

Ex. 8. Find the value of

$$\sin^3 a + \sin^3 3a + \sin^3 5a + \ldots \text{ to } n \text{ terms.}$$

Since $$4\sin^3 a = 3\sin a - \sin 3a,$$

$$\begin{aligned} \therefore\ 4S &= (3\sin a - \sin 3a) + (3\sin 3a - \sin 9a) \\ &\qquad + (3\sin 5a - \sin 15a) + \ldots \\ &= 3(\sin a + \sin 3a + \sin 5a + \ldots) \\ &\quad - (\sin 3a + \sin 9a + \sin 15a + \ldots) \\ &= \frac{3\sin^2 na}{\sin a} - \frac{\sin^2 3na}{\sin 3a}. \end{aligned}$$

EXAMPLES XLV.

Sum the following series to n terms:

1. $\sin 2A + \sin 5A + \sin 8A + \dots$

2. $\cos A + \cos 3A + \cos 5A + \dots$

3. $\cos\frac{A}{3} + \cos\frac{4A}{3} + \cos\frac{7A}{3} + \dots$

4. $\cos\theta + \cos\left(\theta + \frac{\pi}{n}\right) + \cos\left(\theta + \frac{2\pi}{n}\right) + \dots$

5. $\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \dots$

6. $\sin\frac{A}{2} + \sin A + \sin\frac{3A}{2} + \dots$

Find the sum of:

7. $\sin\frac{\pi}{21} + \sin\frac{3\pi}{21} + \sin\frac{5\pi}{21} + \dots + \sin\frac{19\pi}{21}$.

8. $\cos\frac{\pi}{23} + \cos\frac{3\pi}{23} + \cos\frac{5\pi}{23} + \dots + \cos\frac{21\pi}{23}$.

9. $\sin\frac{\pi}{2n-1} + \sin\frac{3\pi}{2n-1} + \sin\frac{5\pi}{2n-1} + \dots$ to n terms.

Find the sum to n terms of:

10. $\sin\alpha - \sin 2\alpha + \sin 3\alpha - \dots$

11. $\cos 2\alpha - \cos 4\alpha + \cos 6\alpha - \dots$

12. $\sin 2\alpha - \sin\left(2\alpha + \frac{\pi}{n}\right) + \sin\left(2\alpha + \frac{2\pi}{n}\right) - \dots$

13. $\cos 3\alpha - \cos\left(3\alpha - \frac{\pi}{n}\right) + \cos\left(3\alpha - \frac{2\pi}{n}\right) - \dots$

14. $\sin\alpha\sin 3\alpha + \sin 3\alpha\sin 5\alpha + \sin 5\alpha\sin 7\alpha + \dots$

15. $\cos\alpha\cos 3\alpha + \cos 3\alpha\cos 5\alpha + \cos 5\alpha\cos 7\alpha + \dots$

16. $\sin\theta\cos 4\theta + \sin 3\theta\cos 6\theta + \sin 5\theta\cos 8\theta + \dots$

17. $\frac{1}{\cos\alpha\cos 3\alpha} + \frac{1}{\cos 3\alpha\cos 5\alpha} + \frac{1}{\cos 5\alpha\cos 7\alpha} + \dots$

18. $\frac{1}{\sin\alpha\sin 4\alpha}+\frac{1}{\sin 4\alpha\sin 7\alpha}+\frac{1}{\sin 7\alpha\sin 10\alpha}+\ldots.$

19. $\sec 2\alpha\sec 4\alpha+\sec 4\alpha\sec 6\alpha+\sec 6\alpha\sec 8\alpha+\ldots.$

20. $\operatorname{cosec} 2\alpha\operatorname{cosec} 3\alpha+\operatorname{cosec} 3\alpha\operatorname{cosec} 4\alpha+\operatorname{cosec} 4\alpha\operatorname{cosec} 5\alpha+\ldots.$

21. $\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\ldots.$

22. $\cos^2 2\alpha+\cos^2 3\alpha+\cos^2 4\alpha+\ldots.$

23. $\sin^2\alpha+\sin^2\left(\alpha+\frac{\pi}{n}\right)+\sin^2\left(\alpha+\frac{2\pi}{n}\right)+\ldots$

24. $\cos^3\alpha+\cos^3(\alpha+\beta)+\cos^3(\alpha+2\beta)+\ldots.$

25. $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\ldots.$

26. $\sin^4\alpha+\sin^4 2\alpha+\sin^4 3\alpha+\ldots.$

27. $\cos^4\alpha+\cos^4 3\alpha+\cos^4 5\alpha+\ldots.$

Find the sum to n terms of:

28. $\sin\alpha+\sin\frac{n-4}{n-2}\alpha+\sin\frac{n-6}{n-2}\alpha+\ldots.$

29. $\frac{1}{\cos\theta+\cos 3\theta}+\frac{1}{\cos\theta+\cos 5\theta}+\frac{1}{\cos\theta+\cos 7\theta}+\ldots.$

30. $\sin\alpha-\sin 2\alpha+\sin 3\alpha-\sin 4\alpha+\ldots.$

31. $\cos\alpha\cos 2\alpha\cos 3\alpha+\cos 2\alpha\cos 3\alpha\cos 4\alpha+\ldots.$

32. $\tan^{-1}\frac{x}{1+1\,.\,2x^2}+\tan^{-1}\frac{x}{1+2\,.\,3x^2}+\ldots.$

33. $\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\ldots.$

Prove that:

34. $\frac{\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots+\sin n\alpha}{\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots+\cos n\alpha}=\tan\frac{n+1}{2}\alpha.$

35. If A_1, $A_2 \ldots A_{2n+1}$ are the angular points of a regular polygon inscribed in a circle and O a point on the arc between A_1 and A_{2n+1}; prove that

$$OA_1+OA_3+\ldots+OA_{2n+1}=OA_2+OA_4+\ldots+OA_{2n}.$$

36. From any point on the circumference of a circle of radius r, chords are drawn to the angular points of the regular inscribed polygon of n sides. Show that the sum of the squares of the chords is $2nr^2$.

CHAPTER XXII.

EXPONENTIAL THEOREM.

213. It is proved in Algebra that

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

is a one-valued, continuous, convergent series for all real values of x; we shall denote it by $\mathsf{E}(x)$.

214. To prove

$$\mathsf{E}(\mathbf{x}) \times \mathsf{E}(\mathbf{y}) = \mathsf{E}(\mathbf{x}+\mathbf{y}).$$

The general term of

$$\mathsf{E}(x) \times \mathsf{E}(y)$$

$$= \frac{x^r}{r!} + \frac{x^{r-1}}{(r-1)!} \cdot \frac{y}{1!} + \frac{x^{r-2}}{(r-2)!} \cdot \frac{y^2}{2!} + \ldots\ldots + \frac{y^r}{r!}$$

$$= \frac{1}{r!}\left[x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \ldots\ldots + y^r\right]$$

$$= \frac{(x+y)^r}{r!}$$ assuming the Binomial Theorem for a positive integral index

$=$ general term of $\mathsf{E}(x+y)$;

$$\therefore\ \mathsf{E}(x) \times \mathsf{E}(y) = \mathsf{E}(x+y).$$

Similarly

$$\mathsf{E}(x) \times \mathsf{E}(y) \times \mathsf{E}(z) \ldots\ldots = \mathsf{E}(x+y+z+\ldots\ldots).$$

215. To prove $\{\mathrm{E}(1)\}^{\mathbf{x}} = \mathrm{E}(\mathbf{x})$.

1st when x is a positive integer.

By Art. 214

$$\begin{aligned}\mathrm{E}(1) \times \mathrm{E}(1) \times \ldots\ldots &\text{ to } x \text{ factors}\\ &= \mathrm{E}(1 + 1 + 1 + \ldots\ldots \text{ to } x \text{ terms})\\ &= \mathrm{E}(x);\end{aligned}$$

$$\therefore \{\mathrm{E}(1)\}^x = \mathrm{E}(x).$$

2nd when x is a positive fraction $= \dfrac{h}{k}$.

By Art. 214

$$\left\{\mathrm{E}\left(\frac{h}{k}\right)\right\}^k = \mathrm{E}\left(\frac{h}{k} + \frac{h}{k} + \ldots\ldots \text{ to } k \text{ terms}\right)$$

$$= \mathrm{E}(h) = \{\mathrm{E}(1)\}^h;$$

$$\therefore \mathrm{E}\left(\frac{h}{k}\right) = \{\mathrm{E}(1)\}^{\frac{h}{k}};$$

$$\therefore \mathrm{E}(x) = \{\mathrm{E}(1)\}^x.$$

3rd when x is negative $= -h$.

Then by Art. 214

$$\mathrm{E}(-h) \times \mathrm{E}(h) = \mathrm{E}(0) = 1;$$

$$\therefore \mathrm{E}(-h) = \frac{1}{\mathrm{E}(h)};$$

$$\therefore \mathrm{E}(x) = \frac{1}{\mathrm{E}(h)} = \frac{1}{\{\mathrm{E}(1)\}^h} = \{\mathrm{E}(1)\}^{-h}$$

$$= \{\mathrm{E}(1)\}^x.$$

216. $\mathrm{E}(1) \equiv 1 + \dfrac{1}{\underline{|1}} + \dfrac{1}{\underline{|2}} + \dfrac{1}{\underline{|3}} + \ldots\ldots$

is generally denoted by e.

Thus $\mathrm{e}^{\mathbf{x}} = \{\mathrm{E}(1)\}^x = \mathrm{E}(x) = 1 + \dfrac{\mathbf{x}}{\underline{|1}} + \dfrac{\mathbf{x}^2}{\underline{|2}} + \dfrac{\mathbf{x}^3}{\underline{|3}} + \ldots\ldots$

This is called the Exponential Theorem and it has been proved for any *real* commensurable exponent.

217. *To prove that e is incommensurable.*

Suppose it is commensurable and equal to $\frac{m}{n}$, m and n being integers, then

$$\frac{m}{n} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots \frac{1}{\lfloor n} + \frac{1}{\lfloor n+1} + \ldots\ldots;$$

multiplying by $\lfloor n$

$$m \lfloor n-1 = \text{a whole number} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots$$

But

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots$$

$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots\ldots$$

$$< \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}}$$

$$< \frac{1}{n};$$

$$\therefore \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots \neq \text{a whole number},$$

thus $m \lfloor n-1 = $ a whole number + a fraction, which is impossible.

$\therefore$ e is incommensurable.

218. Logarithmic series.

$$a^n = e^{\log_e a^n} = e^{n \log_e a}$$

$$= 1 + (n \log_e a) + \frac{(n \log_e a)^2}{\lfloor 2} + \frac{(n \log_e a)^3}{\lfloor 3} + \ldots\ldots$$

Let $a = 1 + x$, x being a proper fraction, positive or negative, then

$$(1+x)^n = 1 + n \log_e (1+x) + \frac{\{n \log_e (1+x)\}^2}{\lfloor 2} + \ldots\ldots;$$

$$\therefore\ 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \ldots\ldots$$

$$= 1 + n \log_e(1+x) + \frac{\{n \log_e(1+x)\}^2}{\lfloor 2} + \ldots\ldots.$$

Both series are convergent and therefore we may equate the coefficients of n;

$$\therefore\ \mathbf{\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots}$$

Changing x into $-x$, we have

$$\mathbf{\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots}$$

These series are convergent when x is limited as above;

$$\therefore\ \log_e \frac{1+x}{1-x} = \log_e(1+x) - \log_e(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right).$$

219. Calculation of Logarithms.

In the above series put $\dfrac{m}{n} = \dfrac{1+x}{1-x}$, then

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\ldots\right\}.$$

Put $\quad m = 2,\ n = 1.$

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \ldots\ldots\right\}$$

$$= 2\left\{\begin{array}{r} \cdot 333333333 \\ \cdot 012345679 \\ 823045 \\ 65324 \\ 5645 \\ 513 \\ 48 \\ \hline \cdot 3465736 \end{array}\right.$$

$= \cdot 693147$ (correct to six places).

Also, by putting $m=3$, $n=2$,

$$\log_e 3 - \log_e 2 = 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \cdot\,\ldots\ldots\right\}$$
$$= \cdot 405465\,;$$
$$\therefore\ \log_e 3 = 1\cdot 09861 \text{ (correct to three places),}$$

and so on.

220. *To find the limiting values of*

$$\left(\cos\frac{\alpha}{n}\right)^n \textit{ and } \left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n$$

when n is indefinitely increased.

Let $$x = \left(\cos\frac{\alpha}{n}\right)^n = \left(1 - \sin^2\frac{\alpha}{n}\right)^{\frac{n}{2}}$$

then $$\log_e x = \frac{n}{2}\log_e\left(1 - \sin^2\frac{\alpha}{n}\right)$$
$$= -\frac{n}{2}\left(\sin^2\frac{\alpha}{n} + \frac{1}{2}\sin^4\frac{\alpha}{n} + \ldots\ldots\right)$$
$$= -\frac{n}{2}\sin\frac{\alpha}{n}\left(\sin\frac{\alpha}{n} + \frac{1}{2}\sin^3\frac{\alpha}{n} + \ldots\ldots\right);$$

now $$\underset{n=\infty}{\text{Lt}}\ \frac{n}{2}\sin\frac{\alpha}{n} = \frac{\alpha}{2} \text{ (Art. 199),}$$

and $$\underset{n=\infty}{\text{Lt}}\left(\sin\frac{\alpha}{n} + \frac{1}{2}\sin^3\frac{\alpha}{n} + \ldots\ldots\right) = 0.$$

Therefore in the limit $\log_e x = 0$, therefore $x = 1$;

$$\therefore\ \underset{n=\infty}{\text{Lt}}\left(\cos\frac{\alpha}{n}\right)^n = 1.$$

Now $$1 > \frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}} > \frac{\sin\frac{\alpha}{n}}{\tan\frac{\alpha}{n}}\left(\text{or } \cos\frac{\alpha}{n}\right) \text{ (Art. 198);}$$

$$\therefore \operatorname*{Lt}_{n=\infty}\left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n \text{ lies between } 1^n \text{ (or 1) and } \operatorname*{Lt}_{n=\infty}\left(\cos\frac{\alpha}{n}\right)^n;$$

$$\therefore \operatorname*{Lt}_{n=\infty}\left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n = 1.$$

EXAMPLES XLVI.

Prove that:

1. $\frac{2}{\underline{|3}} + \frac{4}{\underline{|5}} + \frac{6}{\underline{|7}} + \ldots = e^{-1}.$

2. $\frac{2}{\underline{|1}} + \frac{4}{\underline{|3}} + \frac{6}{\underline{|5}} + \ldots = e.$

3. $e + \frac{1}{e} = 2\left[1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|6}} + \ldots\right].$

4. $e - \frac{1}{e} = 2\left[1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|7}} + \ldots\right].$

5. $\frac{e+1}{e-1} = \frac{1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \ldots}{\frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|6}} + \ldots}.$

6. $5e = 1 + \frac{2^3}{\underline{|2}} + \frac{3^3}{\underline{|3}} + \frac{4^3}{\underline{|4}} + \ldots.$

7. $\operatorname*{Lt}_{n=\infty}\left(1 + \frac{x}{n}\right)^{\frac{n}{y}} = e^{\frac{x}{y}}.$

8. From identity $x = \frac{1 - \frac{1}{x+1}}{1 - \frac{x}{x+1}},$

prove $$\log_e x = \frac{x-1}{x+1} + \frac{x^2-1}{2(x+1)^2} + \frac{x^3-1}{3(x+1)^3} + \ldots.$$

9. From identity $\left(\frac{x+y}{x-y}\right)^2 = \dfrac{1+\dfrac{2xy}{x^2+y^2}}{1-\dfrac{2xy}{x^2+y^2}}$,

prove that $$\log_e \frac{x+y}{x-y} = \frac{2xy}{x^2+y^2} + \frac{1}{3}\left(\frac{2xy}{x^2+y^2}\right)^3 + \frac{1}{5}\left(\frac{2xy}{x^2+y^2}\right)^5 + \dots$$

10. Prove that
$$\log_e (x+1) - \log_e x = 2\left\{\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \dots\right\},$$
and deduce that
$$\log_e 13 = 2\log_e 2 + \log_e 3 + {\cdot}0800427\dots.$$

11. $\log_e \operatorname{cosec}\theta = \frac{1}{2}\cos^2\theta + \frac{1}{4}\cos^4\theta + \frac{1}{6}\cos^6\theta + \dots.$

12. $\log_e \operatorname{cosec}\theta = \frac{1}{2}\cot^2\theta - \frac{1}{4}\cot^4\theta + \frac{1}{6}\cot^6\theta - \dots.$

13. $\frac{1}{2}\log_e \dfrac{\sin\left(\theta+\frac{\pi}{4}\right)}{\sin\left(\theta-\frac{\pi}{4}\right)} = \cot\theta + \frac{1}{3}\cot^3\theta + \frac{1}{5}\cot^5\theta + \dots.$

14. $\sin\theta + \frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + \dots,$
$$= 2\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2} + \frac{1}{5}\tan^5\frac{\theta}{2} + \dots\right),$$
where $$\theta > 0 < \frac{\pi}{2}:$$
use $$\frac{1+\sin\theta}{1-\sin\theta} = \left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)^2.$$

15. $\log_e \sin 2\theta - \log_e \tan\theta = \cos 2\theta - \frac{1}{2}\cos^2 2\theta + \frac{1}{3}\cos^3 3\theta - \dots.$

16. If $a = {\cdot}9999999999$ and $e = 2{\cdot}71828$,

prove that $$a + \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 + \dots = 23{\cdot}02 \text{ approx.}$$

17. Prove that the coefficient of x^n in the expansion of

$$\log_e (1 + x + x^2 + \dots x^{m-1})$$

is either
$$-\frac{m-1}{n} \quad \text{or} \quad \frac{1}{n},$$

according as n is, or is not, a multiple of m.

18. Prove that the limit of $\left(\cos \frac{a}{n}\right)^{n^x}$ when x is an integer and n indefinitely increased is

$$\begin{array}{lll} 0 & \text{when} & x > 2, \\ e^{-\frac{a^2}{2}} & ,, & x = 2, \\ 1 & ,, & x < 2. \end{array}$$

CHAPTER XXIII.

DE MOIVRE'S THEOREM.

221. In this chapter i stands for $\sqrt{-1}$.

Thus $\quad i^2 = -1; \quad i^3 = -i; \quad i^4 = 1;$ etc.

when $$a + ib = a' + ib',$$

a, b, a', b' being real; it is assumed

$$a = a'; \quad b = b'.$$

Such an expression as $a + ib$ is called a complex quantity.

222. De Moivre's theorem. To show that

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, when n is integral,

and that one of the values of

$(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$, when n is fractional.

(i) *When n is a positive integer.*

By actual multiplication

$$\begin{aligned}(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta);\end{aligned}$$

$$\begin{aligned}\therefore\ (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) \\ &= \{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\}(\cos\gamma + i\sin\gamma) \\ &= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)\end{aligned}$$

and so on.

Putting $\alpha = \beta = \gamma = \ldots\ldots = \theta,$

and supposing there are n letters α, β, γ ...... we have

$$\boldsymbol{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.}$$

Thus the theorem is established for a positive integer.

(ii) *When n is a negative integer* $= -m$ suppose

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta.$$

Thus the theorem is established for *any* integer.

(iii) *When n is any fraction* $= \frac{h}{k}$ suppose, h, k being integers.

By (i) and (ii) $\left(\cos\frac{\theta}{k} + i\sin\frac{\theta}{k}\right)^k = \cos\theta + i\sin\theta$ when k is integral;

$\therefore$ $\left(\cos\frac{\theta}{k} + i\sin\frac{\theta}{k}\right)$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{k}}$;

$\therefore$ $\left(\cos\frac{\theta}{k} + i\sin\frac{\theta}{k}\right)^h$ " " " $(\cos\theta + i\sin\theta)^{\frac{h}{k}}$;

i.e. by (i) and (ii),

$\cos\frac{h}{k}\theta + i\sin\frac{h}{k}\theta$ " " " $(\cos\theta + i\sin\theta)^{\frac{h}{k}}$;

i.e. $\cos n\theta + i\sin n\theta$ " " " $(\cos\theta + i\sin\theta)^n$.

Thus the theorem is established for any commensurable number.

223. We have just shown that

$\cos\frac{h}{k}\theta + i\sin\frac{h}{k}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{h}{k}}$;

we now find the others.

Since $(\cos\theta + i\sin\theta)^{\frac{h}{k}} = (\cos h\theta + i\sin h\theta)^{\frac{1}{k}}$,

it follows that we have merely to find the other values of

$$(\cos h\theta + i\sin h\theta)^{\frac{1}{k}}.$$

Putting $h\theta + 2m\pi$ for $h\theta$, where $m =$ a positive integer, we have

$$\cos\left(\frac{h\theta}{k} + \frac{2m\pi}{k}\right) + i\sin\left(\frac{h\theta}{k} + \frac{2m\pi}{k}\right)$$

$$= \text{one of the values of}$$

$$\{\cos(h\theta + 2m\pi) + i\sin(h\theta + 2m\pi)\}^{\frac{1}{k}}$$

$$= \text{one of the values of}$$

$$(\cos h\theta + i\sin h\theta)^{\frac{1}{k}}.$$

Hence by putting

$$m = 0, 1, 2, 3, \ldots\ldots, (k-1),$$

we obtain k values of

$$(\cos h\theta + i\sin h\theta)^{\frac{1}{k}}$$

and these values are all different; for suppose any two are equal,

$$\cos\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right) + i\sin\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right)$$
$$= \cos\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right) + i\sin\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right).$$

Then equating the real and imaginary parts

$$\cos\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right) = \cos\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right)$$

and $$\sin\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right) = \sin\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right);$$

$$\therefore \frac{h\theta}{k} + \frac{2r\pi}{k} - \left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right) = \text{a multiple of } 2\pi,$$

i.e. $$\frac{2\pi}{k}(r - s) = \text{a multiple of } 2\pi,$$

i.e. $$r - s = \quad \text{,,} \quad \text{of } k,$$

which is impossible when both r and s are limited to

$$0, 1, 2, 3, \ldots\ldots, (k-1).$$

Thus we have found k different values of

$$(\cos h\theta + i \sin h\theta)^{\frac{1}{k}},$$

i.e. of $$(\cos\theta + i\sin\theta)^{\frac{h}{k}},$$

and by the Theory of Equations $x^k = c$ has only k roots, *i.e.* no k^{th} root of a quantity can have more than k values.

224. Ex. 1. To extract the nth root of $a + ib$.

1st, put $a + ib$ in the form $r(\cos\theta + i\sin\theta)$.

Thus let $$r\cos\theta = a, \quad r\sin\theta = b;$$

so that $$r^2 = a^2 + b^2, \quad \tan\theta = \frac{b}{a};$$

$$(a + ib)^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right),$$

or $$r^{\frac{1}{n}}\left(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}\right),$$

or $$r^{\frac{1}{n}}\left(\cos\frac{\theta + 4\pi}{n} + i\sin\frac{\theta + 4\pi}{n}\right),$$

....

....

or $$r^{\frac{1}{n}}\cos\left\{\frac{\theta + \overline{2n-2}\pi}{n} + i\sin\frac{\theta + \overline{2n-2}\pi}{n}\right\},$$

and by substituting for r and θ we thus have the n, nth roots of $a + ib$.

Ex. 2. Extract the cube roots of unity.

Here $a=1;\quad b=0;\quad r=1;\quad \theta=0.$

Hence the roots are

$$\cos 0 + i\sin 0;\quad i.e.\ 1.$$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3};\quad i.e.\ \frac{-1+i\sqrt{3}}{2}.$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3};\quad i.e.\ \frac{-1-i\sqrt{3}}{2}.$$

Ex. 3. If $x+\frac{1}{x}=2\cos\mathrm{A}$, prove that $x^n+\frac{1}{x^n}=2\cos n\mathrm{A}$.

Since $$x+\frac{1}{x}=2\cos\mathrm{A};$$

$$\therefore\ x^2-2x\cos\mathrm{A}+1=0.$$

Solving this quadratic in x,

$$x=\cos\mathrm{A}\pm i\sin\mathrm{A}.$$

Taking the positive sign

$$x^n=(\cos\mathrm{A}+i\sin\mathrm{A})^n=\cos n\mathrm{A}+i\sin n\mathrm{A},$$

$$x^{-n}=(\cos\mathrm{A}+i\sin\mathrm{A})^{-n}=\cos n\mathrm{A}-i\sin n\mathrm{A},$$

$$\therefore\ x^n+x^{-n}=2\cos n\mathrm{A}.$$

Similarly for the negative sign.

Ex. 4. Find the value of

$$\frac{(\cos 5\theta+i\sin 5\theta)^2}{(\cos 3\theta-i\sin 3\theta)^3}.$$

$$\text{Exp}^{\text{n}}.=\frac{(\cos\theta+i\sin\theta)^{10}}{(\cos\theta+i\sin\theta)^{-9}}=(\cos\theta+i\sin\theta)^{19}$$

$$=\cos 19\theta+i\sin 19\theta.$$

Ex. 5. If

$$\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0,$$

prove that

$$\cos 4\alpha+\cos 4\beta+\cos 4\gamma$$
$$=2\{\cos 2(\beta+\gamma)+\cos 2(\gamma+\alpha)+\cos 2(\alpha+\beta)\}.$$

Let

$$\cos\alpha+i\sin\alpha=a,\quad \cos\beta+i\sin\beta=b,\quad \cos\gamma+i\sin\gamma=c.$$

Then $$a+b+c=0.$$

Now

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$$
$$\equiv(a+b+c)(a-b-c)(a-b+c)(a+b-c)=0,$$

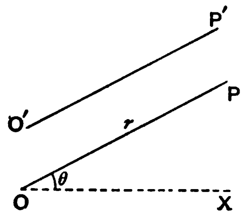
$$\therefore\ \Sigma(\cos\alpha+i\sin\alpha)^4-2\Sigma(\cos\beta+i\sin\beta)^2(\cos\gamma+i\sin\gamma)^2=0,$$
$$\Sigma(\cos 4\alpha+i\sin 4\alpha)-2\Sigma(\cos 2\beta+i\sin 2\beta)(\cos 2\gamma+i\sin 2\gamma)=0,$$
$$\Sigma(\cos 4\alpha+i\sin 4\alpha)-2\Sigma\{\cos 2(\beta+\gamma)+i\sin 2(\beta+\gamma)\}=0.$$

Equating real parts,

$$\Sigma\cos 4\alpha=2\Sigma\cos 2(\beta+\gamma).$$

225. Geometrical representation of complex quantities.

The position of a point P relative to O is defined by the direction and length of the line $\overline{\text{OP}}$.



$\overline{\text{OP}}$ here indicates not merely a line but the operation of moving a point from O to P in the direction of the line OP.

$\overline{\text{OP}}$ is called a Geometrical Vector.

A complex quantity may be represented by a geometrical vector.

The length of OP (r) is called the *Modulus.*

The angle (θ) between OP and a standard direction OX is called the *Amplitude.*

OX is called the *Primary* axis.

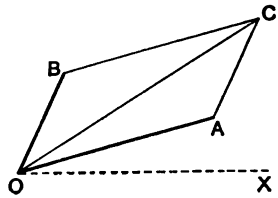
The complex quantity is written (r, θ).

Since we have no conception of absolute position the vector or complex quantity

$$\overline{\text{O}'\text{P}'}\equiv\overline{\text{OP}},$$

when O′P′ is geometrically parallel to OP and equal to it in length, *i.e.* when OP and O′P′ are the opposite sides of a parallelogram.

226. The addition of vectors or complex quantities.



Let $\overline{OA}$ and $\overline{OB}$ be two vectors, complete the parallelogram OACB. Then by Art. 225

$$\overline{OA} + \overline{OB} \equiv \overline{OA} + \overline{AC}$$
$$= \text{the carrying of a tracing point from O to A and then to C}$$
$$= \overline{OC};$$

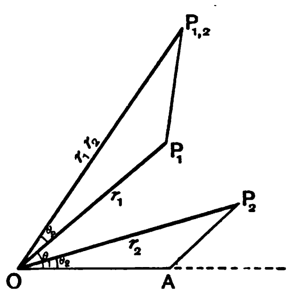
also
$$\overline{OB} + \overline{OA} \equiv \overline{OB} + \overline{BC}$$
$$= \overline{OC},$$
$$\therefore\ \overline{OA} + \overline{OB} = \overline{OB} + \overline{OA}.$$

Thus vectors or complex quantities when added obey the Commutative Law and the sum of any two is represented by the diagonal of a parallelogram having the two as adjacent sides.

By making $A\hat{O}X = B\hat{O}X = n\pi$ (n being any integer) AO and BO become collinear and we obtain the sum of two numbers as arithmetically defined.

227. The multiplication of vectors or complex quantities.

To multiply a by b, we do to a what must be done to unity to obtain b.



Let (r_1, θ_1), (r_2, θ_2) be two complex quantities.

To obtain (r_2, θ_2) from unity we multiply the unit by r_2 and revolve the resulting length through the angle θ_2. [See triangle AOP_2.]

Hence to multiply (r_1, θ_1) by (r_2, θ_2), multiply r_1 by r_2 and thus obtain r_1r_2 for new modulus and then rotate this length from the position θ_1 through an angle θ_2. [See triangle $P_1OP_{1,2}$.] The triangles AOP_2 and $P_1OP_{1,2}$ are seen to be similar;

thus
$$(r_1, \theta_1) \times (r_2, \theta_2) = (r_1r_2, \theta_1 + \theta_2).$$

Again $$r_1 r_2 = r_2 r_1$$
and $$\theta_1 + \theta_2 = \theta_2 + \theta_1;$$

$$\therefore (r_1 r_2,\ \theta_1 + \theta_2) = (r_2 r_1,\ \theta_2 + \theta_1),$$

$$\therefore (r_1,\ \theta_1) \times (r_2,\ \theta_2) = (r_2,\ \theta_2) \times (r_1,\ \theta_1).$$

Thus complex quantities when multiplied obey the Commutative Law.

228. By Art. 227

$$\{r,\ \theta\}^2 = (r^2,\ 2\theta),$$

$$\therefore \text{ one of the values of } \{r^2,\ 2\theta\}^{\frac{1}{2}} \text{ is } (r,\ \theta).$$

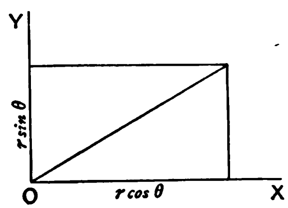
$$\therefore \text{ one of the values of } (1,\ \pi)^{\frac{1}{2}} \text{ is } \left(1,\ \frac{\pi}{2}\right).$$

But $(1,\ \pi)$ is what is usually called -1.

$$\therefore \left(1,\ \frac{\pi}{2}\right) \text{ is what is usually called } \sqrt{-1} \text{ or } i.$$

Thus if OY is perpendicular to OX, unit length along OY represents i.

$\therefore$ a length $r \sin\theta$ along OY represents $i \,.\, r\sin\theta$.



Thus

$$\left(r\sin\theta,\ \frac{\pi}{2}\right) = i \,.\, r\sin\theta,$$

$$(r\cos\theta,\ 0) = r\cos\theta.$$

But by Art. 226

$$(r\cos\theta,\ 0) + \left(r\sin\theta,\ \frac{\pi}{2}\right) = (r, \theta),$$

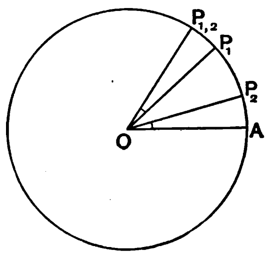
$$\therefore r(\cos\theta + i\sin\theta) = (r,\ \theta).$$

Thus the figure in Art. 227 is the geometrical representation of the identity

$$r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}.$$

229. If in Art. 228 and Fig. Art. 227 we put r_1 and r_2 both equal to unity, we have a geometrical representation of

$$(\cos\theta_1 + i\sin\theta_1) \times (\cos\theta_2 + i\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2),$$



and we see that to multiply one complex quantity by any other, the modulus being unity in both cases, we have merely to rotate the line representing (1, 0) through an angle equal to the sum of the amplitudes of the two quantities; and thus in general, to multiply together any number of complex quantities which have a common modulus unity, we have merely to rotate the line (1, 0) through an angle equal to the sum of the amplitudes of the quantities. And if all the amplitudes are equal we have a geometrical representation of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

i.e. of De Moivre's theorem for a positive index. We thus see also that one of the values of

$$(\cos n\theta + i\sin n\theta)^{\frac{1}{n}}$$

is obtained by rotating the line (1, 0) through an angle $\frac{1}{n}$th of the amplitude of the line whose nth root is indicated.

We can now show how to geometrically represent the other values of

$$(\cos\phi + i\sin\phi)^{\frac{1}{n}}.$$

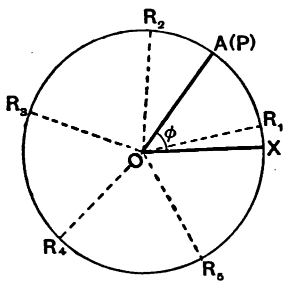
We will do so for the case

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}}$$

the method being perfectly general.

Let a line OP starting from OX, revolve positively. Every time it passes OA it represents

$$(\cos\phi + i\sin\phi).$$



Let OR revolve $\frac{1}{5}$ as fast, then from the above the position of OR at the instant OP passes OA will indicate one value of

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}}.$$

(i) When OP is at OA the 1st time, OR is at OR_1, $X\hat{O}R_1 = \frac{\phi}{5}$, and $(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \cos\frac{\phi}{5} + i\sin\frac{\phi}{5}$;

(ii) when OP is at OA the 2nd time, OR is at OR_2, $XOR_2 = \frac{2\pi+\phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{2\pi+\phi}{5} + i\sin\frac{2\pi+\phi}{5}\right)^{\frac{1}{5}};$$

(iii) when OP is at OA the 3rd time, OR is at OR_3, $XOR_3 = \frac{4\pi+\phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{4\pi+\phi}{5} + i\sin\frac{4\pi+\phi}{5}\right)^{\frac{1}{5}};$$

(iv) when OP is at OA the 4th time, OR is at OR_4, $XOR_4 = \frac{6\pi + \phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{6\pi+\phi}{5} + i\sin\frac{6\pi+\phi}{5}\right)^{\frac{1}{5}};$$

(v) when OP is at OA the 5th time, OR is at OR_5, $XOR_5 = \frac{8\pi + \phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{8\pi+\phi}{5} + i\sin\frac{8\pi+\phi}{5}\right);$$

when OP is at OA the 6th time, OR is at OR_1, the 2nd time;

when OP is at OA the 7th time, OR is at OR_2, the 2nd time;

and so on.

Thus geometrically we get 5 and only 5 different values for

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}}.$$

EXAMPLES XLVII.

Express the following in the form $r(\cos\theta + i\sin\theta)$:

1. $1 + \sqrt{-3}$.
2. $-1 + \sqrt{-3}$.
3. $1 - \sqrt{-3}$.
4. $4 + i.3$.
5. $3 + i.17$.

Find the values of

6. $(-1 + i\sqrt{3})^{\frac{1}{4}}$.
7. $(65 + 142\sqrt{-1})^{\frac{1}{3}}$.
8. $\{(2-\sqrt{3}) + i\}^{\frac{1}{3}}$.
9. Simplify $\dfrac{(\cos\theta + i\sin\theta)^8}{(\cos\phi + i\sin\phi)^9}$.
10. $\dfrac{(\cos\theta - i\sin\theta)^9}{(\cos\phi + i\sin\phi)^7}$.

11. $\dfrac{(\cos 2\theta + i \sin 2\theta)^{-3} (\cos 3\theta - i \sin 3\theta)^{-4}}{(\cos 4\theta - i \sin 4\theta)^{-5} (\cos 5\theta + i \sin 5\theta)^{-6}}$.

12. $\dfrac{(\cos 3\theta - i \sin 3\theta)^{4} (\cos 4\theta + i \sin 4\theta)^{5}}{(\cos 5\theta + i \sin 5\theta)^{6} (\cos 6\theta - i \sin 6\theta)^{7}}$.

13. Prove

$$\sin m\theta = \sec^m \theta \left\{ m \tan \theta - \frac{m(m+1)(m+2)}{\underline{|3}} \tan^3 \theta + \dots \right\}$$

when $$\tan \theta < 1.$$

$$\cos m\theta - i \sin m\theta = (\cos \theta + i \sin \theta)^{-m}.$$

Expand by the Binomial Theorem and equate real and imaginary parts.

14. Show that

$$\left(\cos \frac{\pi}{13} + i \sin \frac{\pi}{13}\right); \quad \left(\cos \frac{3\pi}{13} + i \sin \frac{3\pi}{13}\right); \quad \left(\cos \frac{5\pi}{13} + i \sin \frac{5\pi}{13}\right) \dots$$
$$\left(\cos \frac{11\pi}{13} + i \sin \frac{11\pi}{13}\right);$$
$$\left(\cos \frac{15\pi}{13} + i \sin \frac{15\pi}{13}\right); \dots \left(\cos \frac{25\pi}{13} + i \sin \frac{25\pi}{13}\right)$$

are the roots of

$$x^{12} - x^{11} + x^{10} - x^9 + \dots + 1 = 0.$$

Hence or otherwise show that

$$\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \dots \cos \frac{11\pi}{13} = \frac{1}{2}.$$

15. From the identity

$$\frac{1}{(x_1 - x_2)(x_1 - x_3)} = \frac{1}{(x_2 - x_3)(x_1 - x_2)} - \frac{1}{(x_2 - x_3)(x_1 - x_3)},$$

prove that

$$\sin(\theta_2 + \theta_3) \cos(2\theta_1 + \theta_2 + \theta_3)$$
$$= \sin(\theta_1 - \theta_3) \cos(\theta_1 + 2\theta_2 + \theta_3) - \sin(\theta_1 - \theta_2) \cos(\theta_1 + \theta_2 + 2\theta_3),$$

where $$x_1 = \cos \theta_1 + i \sin \theta_1; \quad x_2 = \text{etc.}$$

16. From the identity

$$\frac{(x_1-x_3)(x_1-x_4)}{(x_2-x_3)(x_2-x_4)}+\frac{(x_1-x_4)(x_1-x_2)}{(x_3-x_4)(x_3-x_2)}+\frac{(x_1-x_2)(x_1-x_3)}{(x_4-x_2)(x_4-x_3)}=1,$$

deduce that

$$\frac{\sin(\theta_1-\theta_3)\sin(\theta_1-\theta_4)}{\sin(\theta_2-\theta_3)\sin(\theta_2-\theta_4)}\sin 2(\theta_1-\theta_2)$$

$$+\frac{\sin(\theta_1-\theta_4)\sin(\theta_1-\theta_2)}{\sin(\theta_3-\theta_4)\sin(\theta_3-\theta_2)}\sin 2(\theta_1-\theta_3)$$

$$+\frac{\sin(\theta_1-\theta_2)\sin(\theta_1-\theta_3)}{\sin(\theta_4-\theta_2)\sin(\theta_4-\theta_3)}\sin 2(\theta_1-\theta_4)=0,$$

where $x_1=\cos\theta_1+i\sin\theta_1;\quad x_2=\text{etc.}$

17. Prove that

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\frac{1}{2},$$

$$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{1}{2}\sqrt{7}.$$

18. Prove that the continued product of the 4 values of

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} \text{ is } -1.$$

CHAPTER XXIV.

EXPANSIONS FOR SINE AND COSINE OF AN ANGLE IN POWERS OF THE CIRCULAR MEASURE OF THE ANGLE.

230. By De Moivre's Theorem

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + ni\cos^{n-1}\theta\sin\theta$$

$$-\frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta - i\,\frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta$$

$$+ \ldots\ldots;$$

$\therefore$ equating real and imaginary parts

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta + \ldots\ldots\ldots \quad \text{(i)},$$

and $$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta$$

$$+ \ldots\ldots.$$

Put $n\theta = \alpha$ and therefore $n = \frac{\alpha}{\theta}$.

Series (i) becomes

$$\cos\alpha = \cos^n\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta} - 1\right)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta + \ldots\ldots$$

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{\underline{|2}}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \ldots\ldots \quad \text{(ii)},$$

$$\therefore \frac{(r+1)^{\text{th}}\text{ term}}{r^{\text{th}}\text{ term}}$$

$$= \frac{\dfrac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-\overline{2r-1}\theta)}{\lfloor 2r} \cos^{n-2r}\theta\left(\dfrac{\sin\theta}{\theta}\right)^{2r}}{\dfrac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-\overline{2r-3}\theta)}{\lfloor 2r-2} \cos^{n-2r+2}\theta\left(\dfrac{\sin\theta}{\theta}\right)^{2r-2}}$$

$$= \frac{(\alpha-\overline{2r-2}\theta)(\alpha-\overline{2r-1}\theta)}{2r(2r-1)}\left(\frac{\tan\theta}{\theta}\right)^2.$$

If now θ becomes indefinitely small and consequently n indefinitely great, α being constant,

$$\underset{n=\infty}{\text{Lt}} \frac{(r+1)^{\text{th}}\text{ term}}{r^{\text{th}}\text{ term}} = \frac{\alpha^2}{2r(2r-1)}, \quad \text{since } \underset{\theta=0}{\text{Lt}}\left(\frac{\tan\theta}{\theta}\right)^2 = 1,$$

and this limit may be made < 1 by taking r great enough.

Thus series (ii) is convergent since the terms are alternately positive and negative and, after a certain term, each term is greater than the succeeding; moreover

$$\underset{\theta=0}{\text{Lt}}\ r^{\text{th}}\text{ term} = \underset{\theta=0}{\text{Lt}} \frac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-\overline{2r-3}\theta)}{\lfloor 2r-2} \cos^{n-2r+2}\theta\left(\frac{\sin\theta}{\theta}\right)^{2r-2}$$

$$= \frac{\alpha^{2r-2}}{\lfloor 2r-2} \quad \text{(Art. 220)},$$

it therefore follows that

$$\cos\alpha < 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \ldots\ldots + \frac{\alpha^{4q}}{\lfloor 4q}$$

$$> 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \ldots\ldots - \frac{\alpha^{4q-2}}{\lfloor 4q-2};$$

or

$$\cos\alpha = 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \ldots\ldots - \frac{\alpha^{4q-2}}{\lfloor 4q-2} + \epsilon\frac{\alpha^{4q}}{\lfloor 4q},$$

where ϵ is a proper fraction.

If now q becomes indefinitely great, the series becomes an infinite one and since $\underset{q=\infty}{\mathrm{Lt}} \dfrac{a^{4q}}{\underline{|4q}} = 0$*

$$\cos a = 1 - \frac{a^2}{\underline{|2}} + \frac{a^4}{\underline{|4}} - \ldots\ldots \infty.$$

In a similar way it may be proved that

$$\sin a = a - \frac{a^3}{\underline{|3}} + \frac{a^5}{\underline{|5}} - \ldots\ldots \infty.$$

231. It is obvious that each of these series is convergent for the terms are alternately positive and negative, and taking the expansion of $\cos a$ for example, the ratio of the $(r+1)^{\text{th}}$ term to the r^{th} is $\dfrac{a^2}{2r(2r-1)}$ which may be made as small as we please by taking r great enough.

If $a \not> \dfrac{\pi}{4}$ these two series converge very rapidly and five or six terms will give the values of $\sin a$ and $\cos a$ to 7 decimal places.

232. Ex. 1. Calculate to 7 decimal places the value of the sine of an angle whose radian measure is ·5.

$$\sin a = \cdot 5 - \frac{1}{\underline{|3}}(\cdot 5)^3 + \frac{1}{\underline{|5}}(\cdot 5)^5 - \ldots,$$

$$\begin{aligned} \cdot 5 &= \cdot 5, \\ (\cdot 5)^3 &= \cdot 125, \\ (\cdot 5)^5 &= \cdot 03125, \\ (\cdot 5)^7 &= \cdot 0078125, \\ (\cdot 5)^9 &= \cdot 001953125 \text{ etc.} \end{aligned}$$

* Suppose $a < c < 4q$ where c is finite and positive.

Then $$\frac{a^{4q}}{\underline{|4q}} = \frac{a^{c-1}}{\underline{|c-1}} \cdot \frac{a \cdot a \cdot a \ldots\ldots a}{c(c+1)(c+2)\ldots\ldots 4q} < \frac{a^{c-1}}{\underline{|c-1}}\left(\frac{a}{c}\right)^{4q-c+1}.$$

Since $a < c$, $$\underset{q=\infty}{\mathrm{Lt}}\left(\frac{a}{c}\right)^{4q-c+1} = 0.$$

Thence required result follows.

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$\frac{-b^2\sin bx}{-a^2\tan bx}.$

$$\therefore \quad \cdot 5 = \cdot 5,$$

$$\frac{1}{\underline{|5}}(\cdot 5)^5 = \cdot 000260416 \qquad \frac{1}{\underline{|3}}(\cdot 5)^3 = \cdot 02083333\dot{3}$$

$$\frac{1}{\underline{|9}}(\cdot 5)^9 = \underline{\cdot 000000005} \qquad \frac{1}{\underline{|7}}(\cdot 5)^7 = \underline{\cdot 000001550}$$

$$\cdot 500260421 \qquad \cdot 020834883.$$

$$\underline{\cdot 020834883}$$

$$\therefore \sin \cdot 5 = \cdot 4794255$$

Ex. 2. Expand $\sin(x+h)$ in powers of h.

$$\begin{aligned}\sin(x+h) &= \sin x \cos h + \cos x \sin h \\ &= \sin x\left(1 - \frac{h^2}{\underline{|2}} + \frac{h^4}{\underline{|4}} \dots\right) + \cos x\left(h - \frac{h^3}{\underline{|3}} + \frac{h^5}{\underline{|5}} - \dots\right) \\ &= \sin x + h\cos x - \frac{h^2}{\underline{|2}}\sin x - \frac{h^3}{\underline{|3}}\cos x + \dots .\end{aligned}$$

Ex. 3. Find (approx.) the number of radians in θ, if

$$\frac{\sin\theta}{\theta} = \frac{5045}{5046}.$$

Since $\dfrac{\sin\theta}{\theta}$ is nearly 1, θ must be small,

$$\therefore \frac{\sin\theta}{\theta} = \frac{\theta - \dfrac{\theta^3}{\underline{|3}}}{\theta}\text{ (approx.)} = 1 - \frac{\theta^2}{\underline{|3}} = \frac{5045}{5046},$$

$$\therefore \theta^2 = \tfrac{1}{5046} \cdot \underline{|3} = \tfrac{1}{841},$$

$$\therefore \theta = \tfrac{1}{29} \text{ radians.}$$

Ex. 4. Find

$$\operatorname*{Lt}_{\theta=0} \frac{\tan 2\theta - 2\tan\theta}{\theta^3}.$$

$$\begin{aligned}\tan x = \frac{\sin x}{\cos x} &= \left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} \dots\right)\left(1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \dots\right)^{-1} \\ &= \left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}}\right)\left(1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}}\right)^{-1}, \text{ if } x \text{ is small} \\ &= \left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}}\right)\left[1 + \left(\frac{x^2}{\underline{|2}} - \frac{x^4}{\underline{|4}}\right) + \left(\frac{x^2}{\underline{|2}}\right)^2\right] \text{ omitting terms} \\ &\qquad \text{beyond } x^5 \\ &= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5,\end{aligned}$$

$$\therefore \frac{\tan 2\theta - 2\tan\theta}{\theta^3} = \frac{[2\theta + \frac{1}{3}(2\theta)^3 + \frac{2}{15}(2\theta)^5] - 2[\theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5]}{\theta^3}$$

$$= \frac{2\theta^3 + 4\theta^5}{\theta^3} = 2 + 4\theta^2,$$

$$\therefore \underset{\theta=0}{\mathrm{Lt}} \frac{\tan 2\theta - 2\tan\theta}{\theta^3} = 2.$$

EXAMPLES XLVIII.

1. Find to 7 places of decimals the sine and cosine of 1 radian.

2. Expand $\cos(\alpha + h)$ in powers of h.

3. Find the general term in the expansion of $\cos^3\theta$ in powers of θ.

4. Find the number of radians in θ, if

$$\frac{\sin\theta}{\theta} = \frac{2645}{2646}.$$

5. Find the general term in the expansion of $\sin^3\theta\cos\theta$ in powers of θ.

6. Find the limit of $\{\sin(\alpha+\theta) - \sin\alpha\}/\theta$, when $\theta = 0$.

7. Find the limit of $(\sin^2 3\theta - \sin^2\theta)/(\cos 4\theta - \cos\theta)$, when $\theta = 0$.

8. Find the limiting value of $[\sin(\tan x) - \tan(\sin x)]/x^7$, when $x = 0$.

9. Find the limiting value of $\dfrac{5\sin\theta - \sin 5\theta}{\theta(\cos\theta - \cos 5\theta)}$ when $\theta = 0$.

10. Find the limit when $x = 0$ of

$$\frac{a^2\sin ax - b^2\sin bx}{b^2\tan ax - a^2\tan bx}.$$

11. Find the limiting value when $x = 0$ of

$$\frac{e^x - 1 + \log_e(1+x)}{x}.$$

12. If $\phi = \theta - 2e\sin\theta + \frac{3e^2}{4}\sin 2\theta - \frac{e^3}{3}\sin 3\theta$,

prove that

$$\theta = \phi + 2e\sin\phi + \frac{5e^2}{4}\sin 2\phi + \frac{e^3}{12}(13\sin 3\phi - 3\sin\phi),$$

where powers of e higher than the third are neglected.

13. Prove that when $x = \tan 2\theta$ and θ lies between $-\frac{\pi}{8}$ and $\frac{\pi}{8}$,

$$\tan\theta = \frac{x}{2}\left(1 - \frac{x^2}{4} + \frac{x^4}{8} - \frac{5}{64}x^6 + \ldots\right),$$

and that, if powers of x above the 5th are neglected,

$$\sin\theta = \frac{x}{2}\left(1 - \frac{3}{8}x^2 + \frac{31}{128}x^4\right).$$

14. If a, b, c are the sides and $\frac{\pi}{3} + \alpha, \frac{\pi}{3} + \beta, \frac{\pi}{3} + \gamma$ the angles (in circular measure) of a triangle which is very nearly equilateral, so that α, β, γ are very small, prove that approximately

$$a\alpha + b\beta + c\gamma = R(\alpha^2 + \beta^2 + \gamma^2),$$

where R is the radius of the circumscribing circle.

15. Prove that the limit of $\left(\cos\frac{a}{n}\right)^{2n^2}$ is e^{-a^2}, when n is infinite.

16. From the expansion of $\cos\theta$ in terms of θ, prove that

$$\Sigma\frac{(b+c)^{2p}(c+a)^{2q}(a+b)^{2r}}{\underline{|2p}\,\underline{|2q}\,\underline{|2r}} = \frac{a^{2n} + b^{2n} + c^{2n} + (a+b+c)^{2n}}{\underline{|2n}}2^{2n-2},$$

where n is a positive integer, and the summation extends to all positive integral values of p, q, r, including zero, such that

$$p + q + r = n.$$

17. If $\cos z = \cos(z + x)\cos\Delta + \sin(z + x)\sin\Delta\cos h$, where x and Δ are so small that higher powers than their cubes may be neglected, prove that

$$x = \Delta\cos h - \tfrac{1}{2}\Delta^2\cot z\sin^2 h + \tfrac{1}{3}\Delta^3\cos h\sin^2 h.$$

18. Express $\sec\theta$ in powers of θ up to θ^8.

19. If θ and ϕ are small angles, prove approximately that

$$\frac{\theta}{\phi} = \frac{2}{3}\frac{\sin\theta}{\sin\phi} + \frac{1}{3}\frac{\tan\theta}{\tan\phi} - \frac{\theta}{180\phi}(\theta^2 - \phi^2)(9\theta^2 - \phi^2).$$

20. Assuming the expansion of $\sin\theta$ in powers of θ, prove that

$$\theta = \sin\theta + \frac{1}{2}\frac{\sin^3\theta}{\theta} + \frac{1.3}{2.4}\frac{\sin^5\theta}{5} + \ldots.$$

21. If $\sin(30^\circ + \theta) = {\cdot}51$, prove that $\theta = 39'\,50''$ (approx.).

TEST PAPERS.

[*Including Properties of Triangles. Chapters XIII and XIV.*]

XLVI.

1. Prove that the radii of the circles inscribed in the triangle into which ABC is divided by the line which bisects the angle A, are to one another in the ratio

$$\cos\frac{C}{2}\left\{1+\tan\frac{C-B}{4}\right\} : \cos\frac{B}{2}\left\{1+\tan\frac{B-C}{4}\right\}.$$

2. Show that in a triangle

$$a^2\cos 2B + b^2\cos 2A = a^2 + b^2 - 4ab\sin A\sin B.$$

3. Prove

$$\tan^2 A - \tan^2 B = \frac{\cos^2 B - \cos^2 A}{\cos^2 A\cos^2 B}.$$

4. In a triangle ABC, I is the centre of the inscribed triangle, ID is perpendicular to BC, BM and CN are perpendicular to AI; show that the triangles MDB and DNC are equiangular and hence prove geometrically that $bc\sin^2\frac{A}{2} = (s-b)(s-c)$.

5. Show how to construct the triangle ABC when r, R and the angle A are given, and establish the limitation that the ratio of r to R must not be greater than

$$2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right).$$

6. Given that $\tan A$ and $\tan B$ are the roots of

$$x^2+px+q=0,$$

find the value of

$$\sin^2(A+B)+p\,.\sin(A+B)\cos(A+B)+q\cos^2(A+B).$$

7. Given

$$\theta+\phi=\alpha \text{ and } \sin^2\theta-\sin^2\phi=k,$$

prove that

$$\sin(\theta-\phi)=\frac{k}{\sin\alpha}.$$

XLVII.

1. O is a point within a triangle ABC such that

$$CAO=ABO=BCO=\alpha,$$

prove that

$$\cot\alpha=\cot A+\cot B+\cot C.$$

2. Prove that

$$\frac{1}{r}=\frac{\cos\frac{A}{2}}{a\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos\frac{B}{2}}{b\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{c\cos\frac{A}{2}\cos\frac{B}{2}}.$$

3. P, Q, R are three successive milestones on a straight road. A is a point such that $A\hat{P}Q=20°$; $A\hat{R}Q=30°$. Find AP in yards.

4. In a circle 5 metres radius what is the length of the arc which subtends an angle 33° 15′ at the centre? ($\pi=\frac{22}{7}$.)

5. ABCD is a quadrilateral, AB = 147, AC = 136, AD = 98 metres. $B\hat{A}C=22°\,30'$; $C\hat{A}D=34°\,15'$. Find the area.

6. Prove that

(i) $$\tan^4\theta=\frac{2\tan\theta-\sin 2\theta}{2\cot\theta-\sin 2\theta}.$$

(ii) $$\tan\theta+\cot\frac{\theta}{2}-\cot\frac{\theta}{2}\sec\theta=0.$$

7. In a triangle

$$2\cos A + \cos B + \cos C = 2,$$

prove that

$$2a = b + c.$$

XLVIII.

1. If
$$\tan(B - C) = \frac{3\sin 2C}{5 - 3\cos 2C},$$
prove that
$$\tan B = 4\tan C.$$

2. Prove that the medians of a triangle form with the sides they bisect 3 angles such that the sum of their cotangents is zero when the angles are measured in the same sense of rotation.

3. If Q and R are the points of trisection of the side BC of a triangle ABC, prove that

$$\sin B\hat{A}R \,.\, \sin C\hat{A}Q = 4\sin B\hat{A}Q \,.\, \sin C\hat{A}R,$$

and

$$(\cot BAQ + \cot QAR)(\cot CAR + \cot RAQ) = 4\operatorname{cosec}^2 QAR.$$

4. Prove

(i) $$\frac{1 + \tan^2\left(\frac{\pi}{4} - \theta\right)}{1 - \tan^2\left(\frac{\pi}{4} - \theta\right)} = \operatorname{cosec} 2\theta.$$

(ii) $$\frac{1}{2}\left(\cot\frac{\theta}{2} - \tan\frac{\theta}{2}\right) = \cot\theta.$$

5. Solve a triangle, given

$$a = 2143;\ c = 4172;\ A = 25^\circ\, 1'.$$

6. The nautical mile is an arc of the earth's equator which subtends an angle 1′ at the centre; find its length correct to the nearest foot, using

one radian = 206265″; earth's equatorial radius = 20926000 ft.

7. Prove that

$$\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3}.$$

XLIX.

1. Solve a triangle, given

$$a = 9621\,;\ b = 6753\,;\ A = 59°41'.$$

2. Prove that $a \sin\theta + b\cos\theta$ lies between

$$\sqrt{a^2+b^2}\,;\ -\sqrt{a^2+b^2},$$

for all values of θ; also that

$$a\sin^2\theta + 2h\sin\theta\cos\theta + b\cos^2\theta$$

lies between

$$\frac{a+b}{2} + \sqrt{h^2 + \tfrac{1}{4}(a-b)^2},$$

and

$$\frac{a+b}{2} - \sqrt{h^2 + \tfrac{1}{4}(a-b)^2}.$$

3. Prove that

$$\frac{AI}{AI_1} + \frac{BI}{BI_2} + \frac{CI}{CI_3} = 1,$$

where ABC is a triangle and I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles.

4. In any triangle, prove that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r} = \frac{s}{r_1} + \frac{s}{r_2} + \frac{s}{r_3},$$

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \frac{(s-a)(s-b)(s-c)}{r^3} = \frac{s^3}{r_1 r_2 r_3}.$$

5. A quadrilateral of perimeter $2s$ inscribed in a circle has two opposite vertices at the ends of a diameter. If a, b are two sides on the same side of the diameter, show that the area of the quadrilateral is $(s-a)(s-b)$.

6. Prove that

(i) $\cos A - \cos 2A = 6\sin^2\frac{A}{2} - 8\sin^4\frac{A}{2}$.

(ii) $\operatorname{cosec} A + 2\operatorname{cosec} 2A = \sec A \cot\frac{A}{2}$.

7. The sides of a triangle are p, q, and $\sqrt{p^2+pq+q^2}$, find the greatest angle.

L.

1. With the ordinary notation prove

$$\Delta = 2R^2 \sin A \sin B \sin C = r^2 \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

2. If the angle A of a triangle is 60°, prove that

$$(a+b+c)(-a+b+c)-3bc=0.$$

3. In a triangle in which $a+b=2c$, prove that

$$a\cos B - b\cos A = 2a-2b.$$

4. In the side CA of a triangle ABC a point A′ is taken and in CB produced B′ is taken so that A′B and AB′ are parallel, prove

$$\frac{AB^2}{AB'\,.\,A'B} = \frac{\sin A' \sin B'}{\sin A \sin B}.$$

5. Prove that

(i) $$\frac{\sin 7\theta + \sin 5\theta}{\cos 5\theta - \cos 7\theta} = \cot\theta.$$

(ii) $$\cos 7\theta - \cos 13\theta = 2\{\sin 11\theta \sin 2\theta + \sin 7\theta \sin 2\theta + \cos 6\theta \cos\theta - \cos 5\theta\}.$$

6. Given

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{1+\tan\alpha\tan\beta},$$

prove that

$$\sin 2\theta = \frac{\sin 2\alpha + \sin 2\beta}{1+\sin 2\alpha \sin 2\beta}.$$

7. In a triangle ABC, b and c are given and it is known that the height AD = the base BC, prove that

$$\frac{c}{2}(\sqrt{5}-1) < b < \frac{c}{2}(\sqrt{5}+1).$$

LI.

1. Prove that

$$\frac{(\sec\theta-\tan\theta)^2-1}{(\sec\theta+\tan\theta)^2-1}=\frac{\sin\theta-1}{\sin\theta+1}.$$

2. If I be the centre of the circle inscribed in a triangle ABC show that

$$\frac{r}{s}=\frac{\text{AI}\,.\,\text{BI}\,.\,\text{CI}}{abc}.$$

3. Prove that

$$\cos 2a+\cos 4a+\cos 6a+\cos 8a=4\cos 5a\,.\,\cos 2a\,.\,\cos a.$$

4. In any triangle show that

$$\frac{b^2-c^2}{\tan \text{A}}+\frac{c^2-a^2}{\tan \text{B}}+\frac{a^2-b^2}{\tan \text{C}}=0.$$

5. Given that $\cos\theta=\dfrac{\cos\phi-x}{1-x\cos\phi}$,

prove that $\tan\dfrac{\theta}{2}=\tan\dfrac{\phi}{2}\sqrt{\dfrac{1+x}{1-x}}.$

6. If $\tan\theta=\dfrac{b}{a}$, prove that

$$a\cos 2\theta+b\sin 2\theta=a.$$

7. In the triangles ABC, A′B′C′ the angles B and B′ are equal, while the angles A and A′ are supplementary; show that

$$aa'=bb'+cc'.$$

[*Including General Values of Equations and Inverse Functions. Chapters XVI and XVIII.*]

LII.

1. Find the length of an arc on the sea which subtends an angle of one minute at the centre of the earth, supposing the earth a sphere of diameter 7920 miles.

2. Solve $$\sin 2\theta = \cos 3\theta.$$

3. If $A + B + C =$ an odd multiple of π, show that

$$\sin^2 B + \sin^2 C = \sin^2 A + 2\cos A \sin B \sin C.$$

4. ABC is a triangle in a horizontal plane, with a right angle at C, and P is the middle point of AB; a flagstaff is set up at C and it is found that its angles of vertical elevation at A, B and P are α, β, γ; show that

$$\tan^2\gamma = 2\tan\alpha\tan\beta\sin 2A.$$

5. The difference between the perimeters of an inscribed and a circumscribed regular dodecagon equals a; show that the difference between their areas equals

$$\frac{a^2}{192\left(1-\cos\frac{\pi}{12}\right)^2}.$$

6. Solve the equation

$$\tan\theta = \frac{1}{6}\cdot\frac{\sin 2\theta}{\cos 2\theta - \frac{1}{3}}.$$

7. Prove that

$$2\tan^{-1}\tfrac{1}{8} + \tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{1}{5} = 45°.$$

LIII.

1. Solve the equation

$$2\sin^2 x = \cos^2\frac{3x}{2}.$$

2. Prove that

(i) $\tan^{-1} 3 + \tan^{-1} 2 + \tan^{-1} 1 = \pi.$

(ii) $\sin^{-1}\frac{5}{13} + \tan^{-1}\frac{7}{24} = \cos^{-1}\frac{253}{325}.$

3. AB is a horizontal road 1 kilometre long running S.E. from A to B. At A a balloon is observed due E. at an elevation of 58°15′, and at B it is seen in a direction N. 27°12′ E. Find the height of the balloon to the nearest metre.

4. In any triangle, prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}.$$

5. From the top of a vertical tower which stands on a flat plain, a length a of a flagstaff projects, and is inclined at an angle γ to the horizon. At a point on the ground, in the vertical plane containing the tower and flagstaff, the elevations of the top of the tower and of the end of the flagstaff are found to be α, β respectively; prove that the height of the tower is

$$a\sin\alpha\sin(\beta+\gamma)\operatorname{cosec}(\beta-\alpha).$$

6. If P is a point in the side BC of a triangle such that $m\mathrm{BP}=n\mathrm{CP}$, show that

$$mc^2+nb^2=(m+n)\,\mathrm{AP}^2+m\mathrm{BP}^2+n\mathrm{CP}^2,$$

and deduce that

$$\mathrm{AP}^2=\frac{(m+n)(mc^2+nb^2)-mna^2}{(m+n)^2}.$$

7. Prove that

$$\tan\theta+2\tan 2\theta=\cot\theta-4\cot 4\theta.$$

LIV.

1. If $A+B+C=180°$, prove that

$$\sin A\sec\tfrac{1}{2}A+(\sin B+\sin C)\tan\tfrac{1}{2}A=(\sin B-\sin C)\cot\tfrac{1}{4}(B-C).$$

2. Solve the equations

(i) $\cot A-\operatorname{cosec}2A=1$,

(ii) $\cos^3 A\sin 3A+\sin^3 A\cos 3A=\dfrac{3\sqrt{3}}{8}$.

3. Prove that

$$\tan^{-1}x=2\tan^{-1}\left[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x\right].$$

4. From two points A and B 150 metres apart in a horizontal plane, the line joining the foot of a tower, in the same plane, to B subtends an angle of 97° 13′ at A, and that joining the foot of the tower to A subtends 22° 29′ at B. Find the height of the tower, if the angle of elevation at A is 37° 10′. Answer to the nearest decimetre.

5. Find a value of x which satisfies the equation

$$4\cos x + 5\sin x = 5{\cdot}2.$$

6. Find the area of a triangle in which the two sides are 187·5 and 925·8 centimetres, and the included angle 27°15′.

7. If I is the centre of the inscribed circle of a triangle ABC, show that the radius of the circle inscribed in the triangle BIC is

$$\sqrt{2}a\,\frac{\sin\frac{B}{4}\sin\frac{C}{4}}{\cos\frac{A}{4}-\sin\frac{A}{4}}.$$

LV.

1. Solve

$$\tan^{-1}(x-1)+\tan^{-1}(2-x)=2\tan^{-1}\sqrt{3x-x^2-2}.$$

2. In any triangle show that

$$abc(1-2\cos A\cos B\cos C)$$
$$=a^3\cos B\cos C+b^3\cos C\cos A+c^3\cos A\cos B.$$

3. Solve

$$\sin\theta-2\sin 2\theta\cos\theta+\cos 3\theta=\sin 3\theta.$$

4. Show that

$$\cos^{-1}\frac{4}{5}-\sin^{-1}\frac{1}{\sqrt{10}}+\tan^{-1}\frac{1}{2}=45°.$$

5. Draw a line BC and divide it at N so that NC = 2BN; draw AN at right angles to BC and equal to BN; join AB, AC and show that

$$2(\tan A+\tan B+\tan C)+3=0.$$

6. OD, OE and OF are the perpendiculars from a point O to the sides of a triangle, show that

$$\cot ADC+\cot BEA+\cot CFB=0.$$

7. Show that

(i) $\cos\tan^{-1}x=\dfrac{1}{\sqrt{1+x^2}}$.

(ii) $\tan\cos^{-1}x=\dfrac{\sqrt{1-x^2}}{x}$.

LVI.

1. If k is the length of the bisector of the angle A of a triangle ABC, prove that

$$\Delta = \frac{1}{2} k (b + c) \sin \frac{A}{2} = \frac{1}{2} ka \cos \frac{B - C}{2}.$$

2. Find all the values of $\tan \theta$ consistent with

   (i) $\cos 4\theta = \cdot 8049.$

   (ii) $\tan (\pi \cot \theta) = \cot (\pi \tan \theta).$

3. Prove that

$$\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5} = \frac{\pi}{4}.$$

4. Solve

$$\tan^{-1} (ax + b) + \tan^{-1} (ax - b) = \frac{\pi}{4}.$$

5. ABC is an equilateral triangle and P is a point in BC such that $PB = \frac{1}{4} BC$; show that $B\hat{A}P = 13^\circ 54'$.

6. If $A + B + C = 180^\circ$, show that

$$\cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} = 4 \cos \frac{\pi + A}{4} \cos \frac{\pi + B}{4} \cos \frac{\pi - C}{4}.$$

7. With the usual notation of inscribed and escribed circles, show that

$$rr_1 (r_2 - r_3) = (b - c) r_2 r_3 \tan \frac{A}{2}.$$

LVII.

1. Show that in any triangle

   (i) $\cos A + \cos B = \dfrac{a + b}{c} \cdot 2 \sin^2 \dfrac{C}{2}.$

   (ii) $a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc.$

2. Prove that

$$\frac{\operatorname{cosec}\theta\operatorname{cosec}\frac{\phi}{2}-\operatorname{cosec}\phi\operatorname{cosec}\frac{\theta}{2}}{\operatorname{cosec}\theta\operatorname{cosec}\frac{\phi}{2}+\operatorname{cosec}\phi\operatorname{cosec}\frac{\theta}{2}}=\tan\frac{\theta+\phi}{4}\tan\frac{\theta-\phi}{4}.$$

3. ABC is a triangle and P is a point within the angle A, such that A and P are on opposite sides of BC. If CP subtends an angle α at A and β at B, show that

$$\text{PB}\sin(\text{C}-\beta+\alpha)=c\sin(\text{A}-\alpha).$$

4. In a triangle ABC the circum-radius is n times the in-radius, prove that

$$\frac{2abc(n+1)}{n}=a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c).$$

5. If $\qquad \tan(2\alpha-3\beta)=\cot(3\alpha-2\beta),$

and $\qquad \tan(2\alpha+3\beta)=\cot(3\alpha+2\beta),$

show that α and β are both multiples of $\frac{\pi}{10}$.

6. Show that

$$2\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\frac{\pi}{4}-\tan^{-1}\frac{6}{61}.$$

7. Show that

$$\tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)-\tan^{-1}\left(\frac{x-\cos\alpha}{\sin\alpha}\right)=\frac{\pi}{2}-\alpha.$$

MISCELLANEOUS EXAMPLES.

1. Prove that

(i) $\cos^2 A + \cos^2(120° + A) + \cos^2(120° - A) = \frac{3}{2}$,

(ii) $\cos^2(A - 45°) + \cos^2 A + \cos^2(A + 45°) + \cos^2(A + 90°) = 2$.

2. In the ambiguous case of the solution of a triangle when a, b, A are given, prove that

$$\text{(i)} \quad c_1 + c_2 = 2b \cos A,$$

$$\text{(ii)} \quad c_1 \sim c_2 = 2a \cos B.$$

3. The lengths of two adjacent sides of a parallelogram are a and b, and their included angle is α; show that the area of the parallelogram formed by the bisectors of the interior angles is $\frac{1}{2}(a-b)^2 \sin \alpha$.

4. The elevation of the top of a flagstaff on the summit of a hill is observed to be α. When the observer walks a distance a directly towards the hill, the top of the flagstaff is found to have an elevation β, while the elevation of the hill top is γ. Show that the height of the hill is

$$a \sin \alpha \cos \beta \tan \gamma \operatorname{cosec}(\beta - \alpha).$$

5. Prove that

$$\sin 2(B - C) + \sin 2(C - A) + \sin 2(A - B)$$
$$= -4 \sin(B - C) \sin(C - A) \sin(A - B).$$

6. Show that $\quad 5 \sin x + 3 \sin(x + 60°) \not> 7.$

7. Solve $\quad \cos 6\theta + \cos 4\theta = \sin 3\theta + \sin \theta.$

8. Prove that

$$\sin 5\theta = 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta.$$

9. If $y = a \sin x + b \cos x$, express y in the form $\text{A} \sin (x + \alpha)$, where A, α are independent of x; and hence show that y must lie in value between $\pm (a^2 + b^2)^{\frac{1}{2}}$.

10. Prove that

$$abc\,(a \cos \text{A} + b \cos \text{B} + c \cos \text{C}) = 8\Delta^2.$$

11. A man travelling along a straight road on a plane observes the angle of elevation of the top of a hill as he passes three successive kilometre-stones to be α, α, β respectively. Prove that the height of the hill is

$$1000\,\{2 \operatorname{cosec} (\alpha + \beta) \operatorname{cosec} (\alpha - \beta)\}^{\frac{1}{2}} \sin \alpha \sin \beta \text{ metres.}$$

12. Prove that

$$\sec \text{A} = \frac{\cos \frac{1}{2}\text{A}}{\sqrt{1 + \sin \text{A}}} + \frac{\sin \frac{1}{2}\text{A}}{\sqrt{1 - \sin \text{A}}}.$$

13. In any triangle prove that

$$\frac{\cos \text{A}}{c \sin \text{B}} + \frac{\cos \text{B}}{a \sin \text{C}} + \frac{\cos \text{C}}{b \sin \text{A}} = \frac{1}{\text{R}}.$$

14. If $\quad \tan (\theta + \alpha) - \tan (\theta - \alpha) = \dfrac{2 \tan \theta}{\cos^2 \theta - \sin^2 \theta \tan^2 \alpha},$

prove that $\quad \theta = \frac{1}{2} n\pi + \alpha \text{ or } (m + \frac{1}{2})\,\pi - \alpha.$

15. If α, β are two angles, not differing by 0 or a multiple of 2π, which satisfy the equation $a \cos x + b \sin x = 1$, then will

$$a = \cos \tfrac{1}{2} (\alpha + \beta) \sec \tfrac{1}{2} (\alpha - \beta), \quad b = \sin \tfrac{1}{2} (\alpha + \beta) \sec \tfrac{1}{2} (\alpha - \beta).$$

16. If P is a point in the side BC of a triangle, such that $m\,.\,\text{BP} = n\,.\,\text{CP}$, show that

$$\frac{\sin \text{B}\hat{\text{A}}\text{P}}{\sin \text{C}\hat{\text{A}}\text{P}} = \frac{nb}{mc}.$$

17. Prove that

$$\cot \left(\frac{\pi}{4} + \theta\right) = \sec 2\theta - \tan 2\theta.$$

18. Show that $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ have the same sign only when $\sin\alpha$ numerically exceeds $\sin\beta$.

19. In any triangle, prove that

$$\tan(A-45°)+\tan(B-60°)+\tan(C-75°)$$
$$=\tan(A-45°)\tan(B-60°)\tan(C-75°).$$

20. Prove that

$$8\sin 10°\sin 40°\sin 80°=2\cos 20°-1.$$

21. In any triangle, prove that

$$a\cos(B-C)+b\cos(C+A)+c\cos(A+B)=0.$$

22. If $\cos\theta+\cos\phi=a$ and $\sin\theta+\sin\phi=b$, prove that

$$\sin(\theta+\phi)=\frac{2ab}{a^2+b^2}.$$

23. From a window on one side of a street, a building on the other side is observed to subtend an angle α. If the width of the street be a feet, and if the height of the point of observation be h feet, show that the height of the building is

$$\frac{(a^2+h^2)\sin\alpha}{a\cos\alpha+h\sin\alpha}.$$

24. Solve $\quad \sin 2\theta+\cos 2\theta=\sin\theta-\cos\theta.$

25. In any triangle, if

$$\cos\theta=\frac{a}{b+c},\quad \cos\phi=\frac{b}{c+a},\quad \cos\psi=\frac{c}{a+b},$$

prove $\quad \tan\frac{1}{2}\theta\tan\frac{1}{2}\phi\tan\frac{1}{2}\psi=\pm\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C.$

26. Prove that

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{3\pi}{9}\tan\frac{4\pi}{9}=3.$$

27. Prove that

$$\tan(45°-\theta)\sin 4\theta=[\cos 2\theta+\sin 2\theta-1][\cos 2\theta-\sin 2\theta+1].$$

28. Two chords of a circle, subtending angles 2α, 2β at the centre O, intersect in a point E within the circle; prove that if θ be the angle between them, and r the radius of the circle, the distance $OE=r\operatorname{cosec}\theta(\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\theta)^{\frac{1}{2}}$, it being supposed that the centre is not within the θ angular space.

29. Two points A, B are observed from points C, D in the same plane, the distance CD being a. The angles ACD, BCD, ADC, BDC are respectively α, β, γ, δ and $\alpha+\gamma=\beta+\delta$. Show that

$$\mathrm{AB}=\frac{a\sin(\alpha-\beta)}{\sin(\alpha+\gamma)}.$$

30. If $$\cos\alpha(1-\sin\beta\sin\gamma)=\cos\beta\cos\gamma,$$
show that
$$1-\sec^2\alpha-\sec^2\beta-\sec^2\gamma+2\sec\alpha\sec\beta\sec\gamma=0.$$

31. Show that the distances x, y, z of the centre of the in-circle from the angular points of a triangle, are connected by the equation
$$ax^2+by^2+cz^2=abc.$$

32. Prove that
$$\sin 3\theta+\sin 5\theta=8\sin\theta\cos^2\theta\cos 2\theta.$$

33. In any triangle, prove that
$$\cos\frac{\mathrm{A}}{2}+\sin\frac{\mathrm{B}}{2}-\sin\frac{\mathrm{C}}{2}=4\cos\frac{\pi-\mathrm{A}}{4}\sin\frac{\mathrm{B}}{4}\cos\frac{\mathrm{C}}{4}.$$

34. If $\mathrm{A}+\mathrm{B}+\mathrm{C}=0$, then
$$\frac{1+\tan\mathrm{A}\tan\mathrm{B}\tan(\mathrm{C}+\mathrm{D})\tan\mathrm{D}}{1-\tan\mathrm{A}\tan\mathrm{B}\tan(\mathrm{C}-\mathrm{D})\tan\mathrm{D}}$$
is unaltered if C is interchanged with A or B.

35. Prove that

(i) $\tan 4\theta=\dfrac{4(\tan\theta-\tan^3\theta)}{1-6\tan^2\theta+\tan^4\theta}$,

(ii) $\cos 2\theta\cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)=\cos 2(\theta+\phi)$.

36. Find $\tan\theta$, where $420\operatorname{cosec}\theta=1369\cos\theta$ and $\theta<\dfrac{\pi}{4}$.

37. Prove that
$$\cot\tfrac{1}{4}\theta-\tan\tfrac{1}{4}\theta=2\cot\theta+2\sqrt{1+\cot^2\theta}.$$

38. The top of a flagstaff is observed from 3 points A, B, C in a straight horizontal line, and the tangents of the elevations are found to be in the ratios $6:3:2$. Show that the distance of the flagstaff from A is

$$\left\{\frac{\text{AB}\,.\,\text{BC}\,.\,\text{AC}}{5\text{AB}-3\text{BC}}\right\}^{\frac{1}{2}}.$$

39. From the vertices A, B, C of the triangle, lines are drawn through the centre of the circumscribing circle, radius R, meeting the circumference again in A′, B′, C′ respectively. Show that the perimeter of the hexagon AB′CA′BC′ is

$$4\text{R}(\cos\text{A}+\cos\text{B}+\cos\text{C}).$$

40. Solve the equation

$$\cos 2x-\sin 2x=\cos x-\sin x-1.$$

41. Prove that

$$\frac{\tan\frac{\text{A}-\text{B}}{2}}{\tan\frac{\text{A}+\text{B}}{2}}+\frac{\tan\frac{\text{A}+\text{B}}{2}}{\tan\frac{\text{A}-\text{B}}{2}}=2\left[\frac{\sin^2\text{A}+\sin^2\text{B}}{\sin^2\text{A}-\sin^2\text{B}}\right].$$

42. If $\cos x=\cos y\cos z+\sin y\sin z\cos\theta,$

then

$$\tan^2\frac{\theta}{2}=\frac{\sin\frac{1}{2}(z+x-y)\sin\frac{1}{2}(x+y-z)}{\sin\frac{1}{2}(x+y+z)\sin\frac{1}{2}(y+z-x)}.$$

43. Perpendiculars p, q are let fall from a point P on the sides CA, CB of a triangle. Show that

$$\text{CP}^2=(p^2+2pq\cos\text{C}+q^2)\operatorname{cosec}^2\text{C}.$$

44. If $\text{A}+\text{B}+\text{C}=180°$, prove that

$$\sin^3\text{A}\sin(\text{B}-\text{C})+\sin^3\text{B}\sin(\text{C}-\text{A})+\sin^3\text{C}\sin(\text{A}-\text{B})=0,$$

and

$$\sin^2\text{A}\sin(\text{B}-\text{C})+\sin^2\text{B}\sin(\text{C}-\text{A})+\sin^2\text{C}\sin(\text{A}-\text{B})$$

cannot vanish unless two of the angles are equal to one another.

45. The adjacent sides of a parallelogram measure a centimetres and b centimetres, and contain an angle β. Prove that the angle at which the diagonals intersect is given by

$$\cos\theta = \pm\frac{a^2-b^2}{\sqrt{a^4-2a^2b^2\cos 2\beta+b^4}}.$$

46. Prove that

$$\operatorname{cosec}^6 A - 1 = \cot^2 A(\cot^4 A + 3\cot^2 A + 3).$$

47. In any triangle, prove that

$$c^2 - 2ac\cos\left(B+\frac{\pi}{3}\right) = b^2 - 2ab\cos\left(C+\frac{\pi}{3}\right).$$

48. Prove that

$$\tan^2\theta + \tan^2\left(\theta+\frac{\pi}{3}\right) + \tan^2\left(\theta+\frac{2\pi}{3}\right) = 9\tan^2 3\theta + 6.$$

49. Prove that

$$3\tan a - 2\cot a = \operatorname{cosec} 2a - 5\cot 2a.$$

50. If C be the mid-point of an arc AB of a circle, centre O, and if OC cut the chord AB in D, show that the area of the segment ACB of the circle is $R^2(\theta - \sin\theta\cos\theta)$ where $\operatorname{vers}\theta = \frac{CD}{R}$, and R is the radius of the circle.

51. In any triangle, prove that

$$(a+b-2c)^2\sec^2\frac{C}{2} + (a-b)^2\operatorname{cosec}^2\frac{C}{2}$$
$$= (b+c-2a)^2\sec^2\frac{A}{2} + (b-c)^2\operatorname{cosec}^2\frac{A}{2}$$
$$= (c+a-2b)^2\sec^2\frac{B}{2} + (c-a)^2\operatorname{cosec}^2\frac{B}{2} = 16(R^2 - 2Rr).$$

52. Solve the equation

$$\sec 4\theta - \sec 2\theta = 2.$$

53. If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in A.P., prove that

$$4(1-\cos\theta)(1-\cos\phi) = \cos\theta + \cos\phi.$$

54. If

$$\frac{1+2\cos(B-C)}{1+\cos B+\cos C}=\frac{1+2\cos(C-A)}{1+\cos C+\cos A}=\frac{1+2\cos(A-B)}{1+\cos A+\cos B},$$

when no two of the angles A, B, C differ by a multiple of 360°, then

$$\sin A+\sin B+\sin C=0,$$

and either $\cos A+\cos B+\cos C=0$

or $\cos A+\cos B+\cos C=-2.$

55. The roof of a barn is in the shape of two similar and equal rectangles inclined at an angle β to the horizon. A person standing opposite one of the side walls at a distance b from it, finds that his eye is in the plane of the roof on that side; when he increases his distance from the wall by c, he finds that the elevation of the top of the roof is γ. Prove that the width of the barn is

$$2\left[c\cos\beta\sin\gamma\operatorname{cosec}(\beta-\gamma)-b\right].$$

56. Solve the equation

$$a\cos\theta+b\sin\theta=c.$$

57. In any triangle, prove that

$$\tan B\tan C+\tan C\tan A+\tan A\tan B=1+\sec A\sec B\sec C.$$

58. In any triangle, show that

$$\frac{r_2+r_3}{(s-a)\sin A}=\frac{r_3+r_1}{(s-b)\sin B}=\frac{r_1+r_2}{(s-c)\sin C}=\frac{abcs}{2\Delta^2}.$$

59. Find in degrees the sum of the three acute angles,

$$\sin^{-1}\tfrac{12}{13},\quad \cos^{-1}\tfrac{7}{25},\quad \tan^{-1}\tfrac{457}{49}.$$

60. The sides of a square, taken in order, subtend angles $\alpha, \beta, \gamma, \delta$ at an internal point: prove that

$$\frac{1}{\cot\alpha+\cot\gamma}+\frac{1}{\cot\beta+\cot\delta}=1.$$

61. Prove that

$$\tan 82\tfrac{1}{2}^\circ=\sqrt{6}+\sqrt{3}+\sqrt{2}+2.$$

62. If $A+B+C=90^\circ$, prove that

$$\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C - \cot B \tan C - \cot C \tan B - \cot C \tan A - \cot A \tan C - \cot A \tan B - \cot B \tan A = 2.$$

63. If

$$\sin A = p \sin B, \quad \cos A = q \cos B, \quad \sin A + \cos A = r(\sin B + \cos B),$$

prove that

$$(p-r)^2(1-q^2)+(q-r)^2(1-p^2)=0.$$

64. ACBP is a quadrilateral figure such that the angle APB (2β) is bisected by the diagonal CP. If $CA=a$, $CB=b$, and the angle $ACB=\alpha$, prove that

$$CP = \frac{ab}{\sin\beta} \cdot \frac{\sin(\alpha+2\beta)}{\sqrt{a^2+b^2+2ab\cos(\alpha+2\beta)}}.$$

65. In a four-sided field ABCD, the angles subtended by BC, DC at A are respectively 60° and 30°; the angles subtended by AD, DC at B are respectively 30° and 60°; and the length of AB is 300 feet. Find the length of CD and the area of the field.

66. Prove that

$$4\cos\theta\cos(120^\circ-\theta)\cos(120^\circ+\theta)=\cos 3\theta.$$

67. Solve

(i) $a(\cos\theta-\cos 2\theta)=b(\sin\theta-\sin 2\theta)$.

(ii) $\sin x+\sin 3x=\cos 2x+\cos 4x$.

68. In any triangle, prove that

$$\cos^2(A-B)+\cos^2(A-C)+2\cos(A-B)\cos(A-C)\cos A = (1+8\sin B\sin C\cos A)\sin^2 A.$$

69. Show that if $\Sigma\cos(\beta-\gamma)=-\frac{3}{2}$, then

$$\cos^3(\alpha+\theta)+\cos^3(\beta+\theta)+\cos^3(\gamma+\theta) - 3\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta)$$

vanishes whatever be the value of θ.

70. Lines are drawn within a triangle ABC through the vertices A, B, C making the same angle θ with the sides AB, BC, CA respectively. Prove that the area of the triangle formed by these lines is to the area of the given triangle as

$$(\cot\theta-\cot A-\cot B-\cot C)^2 : \operatorname{cosec}^2\theta.$$

71. A statue 30 feet high, standing on the top of a tower, subtends at a point, distant 150 feet in a horizontal line from the base of the tower, the same angle as that subtended at the same point by a man 6 feet high standing at the base; find (to $\frac{1}{10}$ of a foot) the height of the tower.

72. Prove that

$$4(\sin 24° + \cos 6°) = \sqrt{3} + \sqrt{15}.$$

73. If a triangle ABC be in a horizontal plane, and an object, P, vertically above A, have angles of elevation of 60°, 45°, and 30° at B, M, and C respectively, show that AP is equal to $\frac{a\sqrt{6}}{4}$, M being the middle point of BC.

74. P is a point inside a triangle ABC at distances x, y, z from the vertices A, B, C respectively; if α, β, γ be the angles subtended at P by the sides a, b, c, show that

$$\frac{ax}{\sin(\alpha - A)} = \frac{by}{\sin(\beta - B)} = \frac{cz}{\sin(\gamma - C)} = \frac{abc}{x\sin\alpha + y\sin\beta + z\sin\gamma}.$$

75. If the tangents of the angles of a triangle are in arithmetical progression, show that the squares of the sides are in the ratios

$$x^2(x^2+9) \ : \ (3+x^2)^2 \ : \ 9(1+x^2)$$

where x is the least or greatest tangent.

76. Prove that

(i) $$\operatorname{cosec}\frac{\pi}{2} + \operatorname{cosec}\frac{\pi}{4} + \operatorname{cosec}\frac{\pi}{8} = \cot\frac{\pi}{16}.$$

(ii) $$\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}\left(\operatorname{cosec}\frac{\pi}{14} - 1\right).$$

77. If $p = 1 + \sin^2\theta$ and $q = 1 + \cos^2\theta$, show that

$$2(p^3 + q^3) + 9q^2 = 27(1 + \cos^4\theta).$$

78. Solve $\cos x - \sin x = \cos\alpha + \sin\alpha$.

79. In any triangle, prove that

$$(b^2 - c^2)\cot^2\frac{A}{2} + (c^2 - a^2)\cot^2\frac{B}{2} + (a^2 - b^2)\cot^2\frac{C}{2}$$
$$= -\frac{1}{r^2}(a+b+c)(b-c)(c-a)(a-b).$$

80. Prove that

$$\{\cos(\sin^{-1} x)\}^2 = \{\sin(\cos^{-1} x)\}^2.$$

81. If $\tan 3A + \tan 2A = 0$, show that $\tan A$ may have any one of the values

$$0, \ \pm\sqrt{5 \pm 2\sqrt{5}}.$$

82. The distance between the centres of two wheels is a, and the sum of their radii is c, show that the length of the string which crosses between the wheels and just wraps around them is

$$2\left\{\sqrt{a^2 - c^2} + c\cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

83. A hexagon, two of whose sides are of length a, two of length b, and two of length c, is inscribed in a circle of diameter d. Prove that

$$d^3 = (a^2 + b^2 + c^2)\,d + 2abc.$$

84. In any triangle, prove that

$$a^3 \cos B \cos C + b^3 \cos C \cos A + c^3 \cos A \cos B$$
$$= abc\,(1 - 2\cos A \cos B \cos C).$$

85. Solve the equation

$$\cos 3x \cos\beta + \sin\alpha \sin\gamma = \cos(3x - \alpha)\cos(3x - \gamma).$$

86. Prove that

$$\sin 20^\circ + \sin 50^\circ + \sin 70^\circ = 4\cos 10^\circ \cos 25^\circ \cos 55^\circ.$$

87. Prove that

$$\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ = 1 + \sin^2 9^\circ + \sin^2 18^\circ.$$

88. If $A + B + C = \pi$, prove that

$$\Sigma \cos^4 A + \Sigma \sin^4 A = 2 + \cos 2A \cos 2B \cos 2C.$$

89. If

$$\sin(\alpha + \beta + \gamma) - \cos(\alpha + \beta + \gamma) + 2\sin\alpha \sin\beta \sin\gamma$$
$$+ 2\cos\alpha \cos\beta \cos\gamma = 0,$$

then either α, β, or γ is of the form $n\pi - \frac{\pi}{4}$.

90. If in the 'Ambiguous Case' of a triangle O_1, O_2; G_1, G_2; P_1, P_2 be respectively the two positions of the circumcentre, centroid and orthocentre, prove that

$$2O_1O_2 = 3G_1G_2 \operatorname{cosec} A = P_1P_2 \sec A,$$

A being the given angle.

91. In a triangle which has $\Sigma \cot A < 2$, show that the least angle $> \cot^{-1} \frac{4}{3}$ and the greatest $< 90°$.

92. DEF is a triangle similar to ABC, and DE is at right angles to BC, while the vertices D, E, F lie on AB, BC, CA respectively. Prove that if a, b, c, are the sides of ABC the circum-radius of DEF is

$$\frac{a^2bc^2}{2a^2c^2 + b^2c^2 + a^2b^2 - b^4}.$$

93. Eliminate ϕ and ϕ' from

$$r = \frac{ab\cos(\theta - \phi)}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} = \frac{ab\cos(\theta - \phi')}{\sqrt{a^2\sin^2\phi' + b^2\cos^2\phi'}},$$

and
$$\tan\phi\tan\phi' = -\frac{b^2}{a^2},$$

and show that
$$2r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

94. In any triangle, prove that

$$a^2\cos^2 A + b^2\cos^2 B + c^2\cos^2 C + 2bc\cos 2A\cos B\cos C$$
$$+ 2ca\cos 2B\cos C\cos A + 2ab\cos 2C\cos A\cos B = 0.$$

95. If $A + B + C + D = 0$, prove that

$$\sin(A + B)\sin(A - B) + \sin(C + D)\sin(C - D)$$
$$= 2[\sin B\sin D\cos A\cos C - \sin A\sin C\cos B\cos D].$$

96. ABC is an equilateral triangle, whose side is a, and P any point on the circumference of the inscribed circle; show that

$$PA^2 + PB^2 + PC^2 = \tfrac{5}{4}a^2.$$

97. Prove that the perpendiculars from the vertices of a triangle on a line joining the orthocentre and circumcentre are

$$2R\cos A\sin(B - C)/\lambda, \quad 2R\cos B\sin(C - A)/\lambda,$$
$$2R\cos C\sin(A - B)/\lambda, \quad \text{where } \lambda^2 = 1 - 8\cos A\cos B\cos C.$$

98. A straight line AD is divided into three equal parts at B and C; the angles subtended by AB, BC, CD at any point P are θ, ϕ, ψ; prove that

$$(\cot\theta + \cot\phi)(\cot\psi + \cot\phi) = 4\operatorname{cosec}^2\phi.$$

99. From a point P, perpendiculars are drawn to the n sides of a regular polygon inscribed in a circle of radius c. If the sum of the squares of these perpendiculars be nh^2, show that the distance δ of the point P from the centre of the polygon is given by

$$\delta^2 = 2\left(h^2 - c^2\cos^2\frac{\pi}{n}\right).$$

100. Prove that

$$(1 + \sin\theta)(3\sin\theta + 4\cos\theta + 5)$$

is a perfect square.

101. The circumference of a given circle is divided into n equal parts at A, A_1, A_2,......A_{n-1}; if the distances of the points A_1, A_2,......A_{n-1} from A be denoted by a_1, a_2,......a_{n-1}, show that

$$a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-2}a_{n-1} = 2nr^2\cos\frac{\pi}{n}.$$

102. Eliminate θ between

$$\sin\theta + \sin 2\theta = a, \text{ and } \cos\theta + \cos 2\theta = b.$$

103. A, B, C are three mountain peaks and the heights of B and C are known to be h and k respectively. At the lowest peak C, it is observed that the lines CA, CB make angles α, β with a horizontal plane and that the angle between the vertical planes through CB and CA is θ. At B it is observed that the angle between the vertical planes through BA and BC is ϕ. Prove that the height of A is

$$k + (h-k)\frac{\tan\alpha\sin\phi}{\tan\beta\sin(\theta+\phi)}.$$

104. If $\alpha + \beta + \gamma + \delta = 0$, prove that

$$\frac{\tan\alpha + \tan\beta + \tan\gamma + \tan\delta}{\cot\alpha + \cot\beta + \cot\gamma + \cot\delta} = \tan\alpha\tan\beta\tan\gamma\tan\delta.$$

105. If θ_1, θ_2, θ_3, θ_4 are four values of θ not differing by multiples of 2π which satisfy the equation

$$a\sin 2\theta + b\sin\theta + c = 0,$$

prove that

(i) $\Sigma \sin \theta_1 = 0$.

(ii) $$4 \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 (\Sigma \sin \theta_1 \sin \theta_2 + 1) = (\Sigma \sin \theta_1 \sin \theta_2 \sin \theta_3)^2.$$

106. Two triangles ABC, A′B′C′ are such that the sides of the first being a, b, c, the sides of the second are $b+c-a$, $c+a-b$, $a+b-c$; prove that

$$\frac{2+\cos A'(1+\cos A)}{\sin A'(1+\cos A)} = \frac{2+\cos B'(1+\cos B)}{\sin B'(1+\cos B)} = \frac{2+\cos C'(1+\cos C)}{\sin C'(1+\cos C)}.$$

107. If

$$\tan(\phi-\theta) = \frac{k^2 \sin 2\phi}{1+k^2\cos 2\phi}$$

and $$\tan\left(\frac{\pi}{4}-\phi\right) = \sin(\theta-\alpha)\operatorname{cosec}(\theta+\alpha),$$

prove that $$\tan\alpha = \frac{1-k^2}{1+k^2}\tan^2\phi.$$

108. In any triangle, prove that

$$b^2\cos 2B + c^2\cos 2C + 2bc\cos(B-C) = a^2\cos 2(B-C).$$

109. Show that, if the medians BE and CF of a triangle meet at G,

$$\tan BGC = \frac{12\Delta}{b^2+c^2-5a^2}.$$

110. If $$\cos(A+B+C)+\cos(B+C-A) + \cos(C+A-B)+\cos(A+B-C) = 0,$$

show that one of the angles A, B, C must be an odd multiple of a right angle.

111. Prove

$$2\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2} = \cos^{-1}\frac{b+a\cos x}{a+b\cos x}.$$

112. If $\cos^3 2\theta + \cos^2 2\theta + \mu^2\cos 2\theta = \mu^2$, show that

$$\mu\tan^3\theta + \tan^2\theta + \mu\tan\theta = 1.$$

113. If $A+B+C=90^\circ$, then

$$\frac{\cos A+\sin B+\sin C}{\sin A+\cos B+\sin C}=\frac{1-\tan\frac{A}{2}}{1-\tan\frac{B}{2}}.$$

114. If $\cos\phi-\cos\theta=m,$

and $\sin\phi-\sin\theta=n,$

show that $\operatorname{cosec}(\theta+\phi)=-\dfrac{m^2+n^2}{2mn}.$

115. If $2\cos\theta=x+\dfrac{1}{x},$

show that $2\cos^3\theta=x^3+\dfrac{1}{x^3}.$

116. If the bisectors of the angles A, B, C of a triangle ABC meet the opposite sides in D, E, F; prove that

$$\frac{4\,(\text{area of ABC})\times(\text{area of DEF})}{\text{AD . BE . CF}}$$

$$=\text{radius of the circle inscribed in ABC.}$$

117. In a triangle ABC, D is a point in BC such that $BD=2CD$, show that

$$AD=\tfrac{1}{3}\sqrt{6b^2+3c^2-2a^2}.$$

118. The sides of a triangle are in Arithmetical Progression and its area is four-fifths that of an equilateral triangle of the same perimeter; show that the sides of the triangle are as

$$7 : 10 : 13.$$

119. If a straight line of length p bisect the angle A of a triangle ABC and divide the base into two parts of lengths m and n, prove that

$$p^2=bc-mn.$$

120. Show that

$$\tan^{-1}\frac{2a-b}{b\sqrt{3}}+\tan^{-1}\frac{2b-a}{a\sqrt{3}}=\frac{\pi}{3}.$$

121. Solve

$$\frac{(66{\cdot}66)^{\frac{3}{4}} \sin^2 33^\circ \sqrt{\cos 337^\circ}}{({\cdot}0033)^{\frac{2}{3}} x^{\frac{1}{3}}} = \tan^5 57^\circ.$$

122. If O is the centre of the circle described round an acute-angled triangle and AO is produced to meet BC in D, show that

$$\text{OD} = \frac{\text{R} \cos \text{A}}{\cos (\text{B} - \text{C})}.$$

123. If the inscribed circle of a triangle ABC touch the sides BC, CA, AB in D, E, F, prove that $\tan \text{ADB} = \dfrac{2r_1}{b-c}$ where r_1 is the radius of the escribed circle which touches BC.

124. Show that the radius of the circle which touches the sides AB, AC of the triangle ABC and also touches the inscribed circle is

$$r \frac{1 - \sin \frac{\text{A}}{2}}{1 + \sin \frac{\text{A}}{2}}.$$

125. If in the ambiguous case the area of the larger triangle is double that of the smaller, show that the tangent of one of the angles at the base is three times that of the other.

126. Solve

$$(\sin 8^\circ + \cos 8^\circ)^{2x} = 2 \sin 16^\circ (\tan 32^\circ)^x.$$

127. Deduce from De Moivre's Theorem

$$\tan n\theta = \frac{n \tan \theta - \dfrac{n(n-1)(n-2)}{\underline{|3}} \tan^3 \theta + \dots}{1 - \dfrac{n(n-1)}{\underline{|2}} \tan^2 \theta + \dfrac{n(n-1)(n-2)(n-3)}{\underline{|4}} \tan^4 \theta - \dots}.$$

128. If $$\tan [\log (a + ib)] = x + iy,$$
prove that $$2x = (1 - x^2 - y^2) \tan \{\log (a^2 + b^2)\}.$$

129. Prove that

$$\log_e \sqrt{8} = 1 + \frac{1}{3 \,.\, 9} + \frac{1}{5 \,.\, 9^2} + \frac{1}{7 \,.\, 9^3} + \dots$$

130. If
$$y = \frac{x}{\lfloor 1} - \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} - \frac{x^4}{\lfloor 4} + \ldots$$
show that
$$x = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots$$
y being numerically less than unity.

131. Prove that the length of a plane arc of small curvature is approximately
$$\frac{c - 40c' + 256c''}{45},$$
where $c =$ the chord of the arc, $c' =$ the chord of half the arc and $c'' =$ the chord of quarter of the arc.

132. Prove that
$$\sec^2\frac{\pi}{9} + \sec^2\frac{3\pi}{9} + \sec^2\frac{5\pi}{9} + \sec^2\frac{7\pi}{9} = 40.$$

133. Draw on squared paper a graph of $\tan 10x - 2\tan 9x + 1$ for values of x between $0°$ and $9°$, and thus show that the expression vanishes when $x = 5°{\cdot}9$.

134. Prove that the eliminant of
$$\frac{1}{a^2} = \frac{\cos^2\theta}{t^2} + \frac{\sin^2\theta}{t'^2}; \quad \frac{1}{b^2} = \frac{\cos^2\phi}{t^2} + \frac{\sin^2\phi}{t'^2}; \quad t\tan\theta\tan\phi = t',$$
is
$$a^2b^2 - t^2t'^2 = 0.$$

135. Prove that
$$\log_e 5 - \log_e 4 = \frac{1}{5} + \frac{1}{2.5^2} + \frac{1}{3.5^3} + \frac{1}{4.5^4} + \ldots.$$

APPENDIX I.

SEVEN FIGURE LOGARITHMS.

233. FOR some purposes it may be necessary to obtain a more accurate result than is possible with 4 figure logarithms.

When the logarithm of a number between 1 and 100,000 is required, the value may be written down at once from the Tables.

To obtain the *mantissa*, we look for the *first four* significant figures in the first column and passing along the row containing these, take the number in that particular column headed by the *fifth* figure; this gives the last 4 digits of the mantissa, the first 3 digits being obtained from the column headed by 0.

A bar placed over the last 4 digits has the same significance as in Art. 67, and indicates that the first 3 digits are obtained from a succeeding instead of a preceding line.

Ex. 1. Find the logarithm of 46·223.

We firstly look out the row containing 4622 in the first column and in this row select the number headed by the fifth figure 3. This gives for the last 4 figures 8581, and the first 3 are 664. Since there are 2 figures to the left of the decimal point in the original number, it follows that the characteristic is 1,

$$\therefore \log 46{\cdot}223 = 1{\cdot}6648581.$$

No.	0	1	2	3	4	5	6	7	8	9	Diff.
4621	6647360	7454	7548	7642	7736	7830	7924	8018	8111	8205	
22	8299	8393	8487	8581	8675	8769	8863	8957	9051	9145	

234. If the number whose logarithm is required contains more than 5 figures, we have to make use of the *Rule of Proportional Parts*, and the column of Differences on the right of the Table becomes an essential feature. This rule is that for small differences, *the increase in the logarithm of a number is proportional to the increase of the number.*

No.	0	1	2	3	4	5	6	7	8	9	Diff. 93
4671	6694099	4192	4285	4378	4471	4564	4656	4749	4842	4935	1 9 2 19
72	5028	5121	5214	5307	5400	5493	5586	5679	5772	5865	3 28 4 37
73	5958	6051	6144	6237	6330	6422	6515	6608	6701	6794	5 47 6 56
74	6887	6980	7073	7166	7259	7352	7445	7537	7630	7723	7 65 8 74
75	7816	7909	8002	8095	8188	8281	8373	8466	8559	8652	9 84

From the table given above

$$\begin{aligned}\log 46718 &= 4{\cdot}6694842\\ \log 46717 &= 4{\cdot}6694749,\\ \therefore\ \text{difference for } 1 &= {\cdot}0000093.\end{aligned}$$

The above rule gives

$$\begin{array}{lll}\text{diff. for } {\cdot}1 = \frac{1}{10} & \text{diff. for } 1 & = {\cdot}0000009 \text{ (correct to 7 places)},\\ \quad,, \quad {\cdot}2 = \frac{2}{10} & \quad,, & = {\cdot}0000019\\ \quad,, \quad {\cdot}3 = \frac{3}{10} & \quad,, & = {\cdot}0000028 \text{ etc.}\end{array}$$

It will be seen that these terminal figures 9, 19, 28 etc. are the same as the figures in the Difference Column, which may therefore be used in future instead of those obtained from the above calculations.

Ex. 2. Find log 4673·8723.

$$\begin{array}{lr|l}\log 4673{\cdot}8 & =3{\cdot}6696701 & \\ \textit{diff. for } 7 & 65 & \\ \qquad\quad 2 & 1 & 9\\ \qquad\quad 3 & & 28,\end{array}$$

$$\therefore\ \log 4673{\cdot}8723 = 3{\cdot}6696768.$$

Ex. 3. Find x, given $\log x = 3{\cdot}6697402$.

$$\begin{array}{lr}\log x = & 3{\cdot}6697402\\ \log 4674{\cdot}5 = & \underline{3{\cdot}6697352}\\ & 50\\ \textit{diff. for } 5 & \underline{47}\\ & 30\\ \qquad\quad 3 & \underline{28}\end{array}$$

$$\therefore\ x = 4674{\cdot}553 = 4{\cdot}674553 \times 10^3.$$

[We firstly find from the Tables the mantissa next below that given, ·6697352, and noticing that the next mantissa is ·6697445 and the difference between these ·0000093, select the difference column headed by 93.]

Ex. 4. Find the value of $(\cdot 002489775)^{\frac{1}{4}}$.

$$\log x = \tfrac{1}{4} \log \cdot 002489775$$

$$\begin{array}{lrr|l}
\log \cdot 0024897 & = \bar{3}\cdot 3961470 & & \\
\textit{diff. for } 7 & 123 & & \\
\qquad 5 & 8 & & 8,
\end{array}$$

$$\therefore \ \log \cdot 002489775 = \bar{3}\cdot 3961602,$$

$$\begin{aligned}
\therefore \ \log x &= \tfrac{1}{4}(\bar{3}\cdot 3961602) \\
&= \bar{1}\cdot 3490401
\end{aligned}$$

$$\begin{array}{lr}
\log \cdot 22337 & = \bar{1}\cdot 3490248 \\ \hline
 & 153 \\
\textit{diff. for } 7 & 137 \\ \hline
 & 160 \\
\qquad 8 & 156
\end{array}$$

$$\therefore \ x = \cdot 2233778 = 2\cdot 233778 \times 10^{-1}.$$

Ex. 5. Find the value of

$$\left[\frac{(\cdot 02587)^3 \times \sqrt{507\cdot 92}}{83\cdot 97 \times \cdot 52}\right]^{\frac{1}{5}}.$$

$$\log x = \tfrac{1}{5}[3 \log \cdot 02587 + \tfrac{1}{2} \log 507\cdot 92 - \log 83\cdot 97 - \log \cdot 52],$$

$$\begin{array}{rlcrl}
3 \log \cdot 02587 & = \bar{5}\cdot 2383892 & \qquad & \log 83\cdot 97 & = 1\cdot 9241242 \\
\tfrac{1}{2} \log 507\cdot 92 & = 1\cdot 3528976 & & \log \quad \cdot 52 & = \bar{1}\cdot 7160033 \\ \hline
 & \bar{4}\cdot 5912868 & & & 1\cdot 6401275 \\
 & 1\cdot 6401275 & & & \\ \hline
 & 5\overline{)\bar{6}\cdot 9511593} & & & \\
\therefore \ \log x & = \bar{2}\cdot 9902319 & & & \\
\log \cdot 097775 & = \bar{2}\cdot 9902278 & & & \\ \hline
 & 41 & & & \\
\textit{diff. for } 9 & 41 & & &
\end{array}$$

$$\therefore \ x = 9\cdot 77759 \times 10^{-2}.$$

Trigonometrical Ratios.

235. In 7 figure tables the sines and cosines are given for all angles between 0° and 45° at intervals of 1 minute, difference columns being provided for calculating the seconds by means of the Rule of Proportional Parts.

Ratios of angles between 45° and 90° can be found by reading upwards from the bottom of the page.

Ex. 1. Find $\sin 29° \, 1' \, 13''$.

From the tables, $\sin 29° \, 1' = \cdot 4850640$

diff. for $60'' = 2544$

$\therefore$ *increase* for $13'' = 551$

$\therefore \sin 29° \, 1' \, 13'' = \cdot 4851191$.

$$\begin{array}{r} 2544 \\ \underline{13} \\ 2544 \\ 7632 \\ 60\overline{)33072} \\ \hline 551 \end{array}$$

Ex. 2. Find $\cos 29° \, 4' \, 34''$.

From the tables, $\cos 29° \, 4' = \cdot 8740550$

diff. for $60'' = 1413$

$\therefore$ *decrease* for $34'' = 801$

$\therefore \cos 29° \, 4' \, 34'' = \cdot 8739749$.

$$\begin{array}{r} 1413 \\ \underline{17} \\ 1413 \\ 9891 \\ 30\overline{)24021} \\ \hline 801 \end{array}$$

Ex. 3. Find the angle whose cosine is $\cdot 8741742$.

$$\begin{aligned} \text{From tables,} \quad \cos 29° \, 3' &= \cdot 8741963 \\ \cos x &= \cdot 8741742 \\ \therefore \text{diff.} &= 221 \\ \text{Now diff. for } 60'' &= 1413 \\ \therefore \text{req. no. of seconds} &= \frac{60 \times 221}{1413} \\ &= 9 \text{ (nearly)} \end{aligned}$$

$\therefore x = 29° \, 3' \, 9''$ (adding, since $\cos x < \cos 29° \, 3'$, $\therefore x > 29° \, 3'$).

$$\begin{array}{r} 9\cdot 3 \\ 1413\overline{)13260} \\ \underline{12717} \\ 5430 \end{array}$$

NATURAL SINES, COSINES, ETC.

29 Deg.

′	Sine	Diff.	Covers.	Chord	Co-Chord	Vers.	Diff.	Cosine	′
0	4848096		5151904	5007600	1·0150768	1253803		8746197	60
		2544					1411		
1	4850640		5149360	5010416	1·0148261	1255214		8744786	59
		2544					1411		
2	4853184		5146816	5013232	1·0145754	1256625		8743375	58
		2543					1412		
3	4855727		5144273	5016048	1·0143247	1258037		8741963	57
		2543					1413		
4	4858270		5141730	5018864	1·0140740	1259450		8740550	56
		2542					1413		
5	4860812		5139188	5021680	1·0138233	1260863		8739137	55
60	5000000	2519	5000000	5176380	1·0000000	1339746	1454	8660254	0
′	Cosine	Diff.	Vers.	Co-Chord	Chord	Covers.	Diff.	Sine	′

60 Deg.

236. In the case of tangents, cotangents, secants and cosecants, all values are given between 0° and 90° at intervals of 1 minute.

Ex. 4. Find tan 49° 1′ 13″.

From the tables, $\tan 49^\circ 1' = 1{\cdot}1510445$

$\text{diff. for } 60'' = \quad 6765$

$\therefore$ *increase* for $13'' = \quad 1466$

$\therefore \tan 49^\circ 1' 13'' = 1{\cdot}1511911.$

$$\begin{array}{r} 6765 \\ 13 \\ \hline 6765 \\ 20295 \\ 60\overline{)87945} \\ \hline 1466 \end{array}$$

Ex. 5. Find cot 34° 58′ 17″.

From the tables, $\cot 34^\circ 58' = 1{\cdot}4299178$

$\text{diff. for } 60'' = \quad 8852$

$\therefore$ *decrease* for $17'' = \quad 2508$

$\therefore \cot 34^\circ 58' 17'' = 1{\cdot}4296670.$

$$\begin{array}{r} 8852 \\ 17 \\ \hline 8852 \\ 61964 \\ 60\overline{)150484} \\ \hline 2508 \end{array}$$

NATURAL TANGENTS.

′	49°	50°	51°	52°	53°	54°	55°	′
0	1·1503684	1·1917536	1·2348972	1·2799416	1·3270448	1·3763819	1·4281480	60
1	1·1510445	1·1924579	1·2356319	1·2807094	1·3278483	1·3772242	1·4290326	59
2	1·1517210	1·1931626	1·2363672	1·2814776	1·3286524	1·3780672	1·4299178	58
60	1·1917536	1·2348972	1·2799416	1·3270448	1·3763819	1·4281480	1·4825610	0
′	40°	39°	38°	37°	36°	35°	34°	′

NATURAL COTANGENTS.

Logarithmic Sines, Cosines, etc.

237. These values are given for all angles between 0° and 90° at intervals of 1 minute, difference columns being provided for the seconds, and the Rule of Proportional Parts again being used.

Ex. 1. Find $L\sec 33° 1' 19''$.

From the tables, $L\sec 33° 1' = 10{\cdot}0764907$

diff. for $60'' =$ 821

$\therefore$ *increase* for $19'' =$ 260

$\therefore$ $L\sec 33° 1' 19'' = 10{\cdot}0765167$.

$$\begin{array}{r} 821 \\ \underline{19} \\ 821 \\ 7389 \\ 60\overline{)15599} \\ 260 \end{array}$$

Ex. 2. Find x, given that $L\operatorname{cosec} x = 10{\cdot}2636425$.

From the tables, $L\operatorname{cosec} 33° 1' = 10{\cdot}2636968$

$\therefore$ diff. $=$ 543

Now diff. for $60'' =$ 1944

$\therefore$ req. no. of seconds $= \dfrac{60 \times 543}{1944}$

$= 17$ (nearly)

$$\begin{array}{r} 16{\cdot}7 \\ 1944\overline{)32580} \\ 1944 \\ \overline{13140} \\ 11664 \\ \overline{14760} \end{array}$$

$\therefore$ $x = 33° 1' 17''$ (adding, since $L\operatorname{cosec} x < L\operatorname{cosec} 33° 1'$, $\therefore$ $x > 33° 1'$).

LOGARITHMIC SINES, ETC.

33 Deg.

′	Sine	Diff.	Cosec.	Tang.	Diff.	Cotang.	Secant	D.	Cosine	′
0	9·7361088		10·2638912	9·8125174		10·1874826	10·0764086		9·9235914	60
		1944			2765			821		
1	9·7363032		10·2636968	9·8127939		10·1872061	10·0764907		9·9235093	59
		1944			2765			821		
2	9·7364976		10·2635024	9·8130704		10·1869296	10·0765728		9·9234272	58
		1942			2764			822		
′	Cosine	Diff.	Secant	Cotang.	Diff	Tang.	Cosec	D.	Sine	′

56 Deg.

EXAMPLES XLIX.

Find the value of

1. $1278{\cdot}4 \times 9276{\cdot}4 \times {\cdot}80051$.

2. ${\cdot}005271 \times 7{\cdot}329 \times {\cdot}00082795$.

3. $827{\cdot}932 \times 51{\cdot}82 \times {\cdot}0079856$.

4. $\dfrac{87{\cdot}563 \times {\cdot}002897}{12598{\cdot}22}$.

5. $\dfrac{457{\cdot}082 \times {\cdot}002987}{421 \times {\cdot}079825}$.

6. $\dfrac{82957{\cdot}9 \times {\cdot}02981 \times {\cdot}72456}{{\cdot}00052897 \times 82476}$.

7. $\left[\dfrac{52{\cdot}478 \times {\cdot}002497}{\frac{1}{3}\sqrt[5]{{\cdot}0029875}}\right]^3$.

8. $\left[\dfrac{85{\cdot}9781 \times {\cdot}002478 \times \frac{1}{3}({\cdot}8275)}{\sqrt[3]{{\cdot}0893476}}\right]^{\frac{1}{4}}$.

9. $1729{\cdot}5 \sin 18° 17' \times \cos 19° 18'$.

10. ${\cdot}0025879 \tan 42° 15' \times \sec 69° 14'$.

11. $\sin 18° 14' 57'' \times \tan 51° 20' 20''$.

12. $({\cdot}0876)^3 \operatorname{cosec} 55° 17' 16''$.

13. $13{\cdot}8297 \times \sqrt[3]{82{\cdot}0092} \cos 47° 15' 16''$.

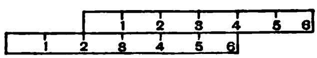
14. $\frac{1}{3} \times {\cdot}0008259 \times \sqrt[3]{825{\cdot}6} \cot 18° 14' 50''$.

15. $\dfrac{{\cdot}02987 \tan 16° 15' 40''}{\sqrt[3]{5298{\cdot}75} \operatorname{cosec} 18° 17' 20''}$.

APPENDIX II.

THE SLIDE RULE.

238. ONE method of adding together lengths is by the use of two rules placed side by side. For instance, if we wished to add 2 and 1, 2 and 2, 2 and 3 etc. we should place them as shown in the diagram, one rule overlapping the other to the extent of 2 divisions; underneath the **1** of the top rule we find the result of $2 + \mathbf{1}$ *i.e.* 3; underneath the **2** of the top rule we find the result of $2 + \mathbf{2}$ *i.e.* 4; underneath the **3** of the top rule we find the result of $2 + \mathbf{3}$ *i.e.* 5, and so on.



In the same way we can subtract. If we wish, for example, to take 3 from 5, we move the rules until the 3 of the top rule coincides with the 5 of the lower one; the result of the subtraction, viz. 2, is then seen under the left-hand end of the top rule.

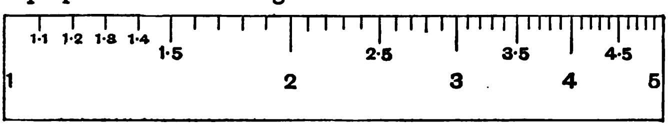
239. *The Slide Rule* is an instrument so graduated that we can perform multiplication and division just as easily as addition and subtraction with ordinary rules. In order to understand the principle on which it works, we merely have to remember that

$$\log abc = \log a + \log b + \log c$$

and
$$\log \frac{a}{b} = \log a - \log b,$$

i.e. in dealing with logarithms, multiplication is replaced by addition and division by subtraction.

240. Let two rules be graduated with unequal divisions so that the distances of any two graduations from the end of the rule are not proportional to the numbers on those graduations, but proportional to the logarithms of the numbers.



The distance from 1 to 3 is not twice the distance from 1 to 2 but

$$\frac{\text{distance from 1 to 3}}{\text{distance from 1 to 2}}=\frac{\log 3}{\log 2}=\frac{\cdot 4771}{\cdot 3010}.$$

Since

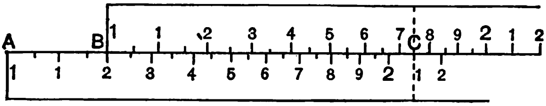
$$\begin{aligned} \log 1 &= 0\\ \log 2 &= \cdot 3010\\ \log 3 &= \cdot 4771\\ \log 4 &= \cdot 6021\\ \log 5 &= \cdot 6990\\ \log 6 &= \cdot 7782\\ \log 7 &= \cdot 8451\\ \log 8 &= \cdot 9031\\ \log 9 &= \cdot 9542\\ \log 10 &= 1\cdot 0000 \end{aligned}$$

it follows that the distances of the graduations 1, 2, 3 ...... 10 from the left-hand end of the rule are proportional to the numbers in the 2nd column, so that 1 is placed at the left-hand end and not 0.

Intermediate graduations are obtained by a similar process.

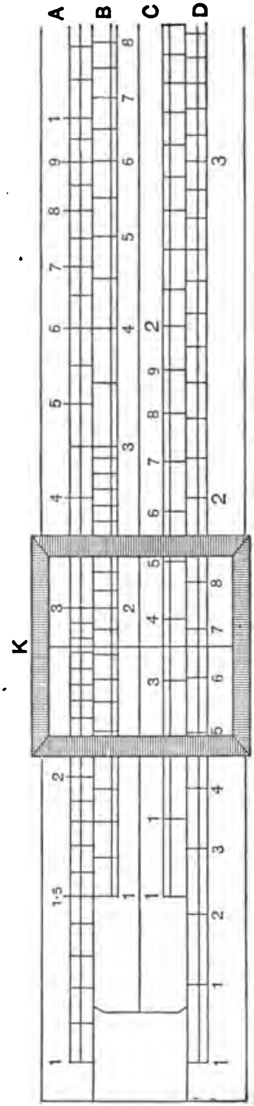
241. Suppose we now wish to use two such rules in order to find the value of $1\cdot 2 \times 1\cdot 75$. One of them is moved until the graduation 1—called the *Index*—is over $1\cdot 2$ of the lower rule; then looking under $1\cdot 75$ of the upper rule we find the product $2\cdot 1$ on the lower rule. The reason for this is that

$$\begin{aligned} \log 1\cdot 2 &= \mathrm{AB}\\ \log 1\cdot 75 &= \mathrm{BC},\\ \therefore\ \log(1\cdot 2\times 1\cdot 75) &= \log 1\cdot 2+\log 1\cdot 75\\ &= \mathrm{AB}+\mathrm{BC}\\ &= \mathrm{AC}\\ &= \log 2\cdot 1;\\ \therefore\ 1\cdot 2\times 1\cdot 75 &= 2\cdot 1. \end{aligned}$$



Similarly if we wish to find the value of $\frac{2\cdot 1}{1\cdot 75}$, the top rule is moved until $1\cdot 75$ on it coincides with $2\cdot 1$ on the lower rule, the

quotient 1·2 is then read off on the lower rule immediately under the Index of the top rule.



242. One extremity of a Slide Rule with some of the graduations marked is shown in the diagram. It will be noticed that there are four scales; A and D being on the Rule and B and C on the Slide. Moreover A and B are graduated in the same way, and C and D in the same way, the distance between any two numbers on C or D being twice as great as that between the corresponding numbers on A or B. *The Cursor*, K, is a rectangular frame with a glass front on which is engraved a black line at right angles to the length of the Rule; the frame is made to slide in grooves.

243. *Multiplication.*

Ex. 1. Find the value of $1·96 \times 1·74$.

Place the index (the 1) of the C scale over 1·96 on the D scale.

The product 3·41 is then read off on the D scale under 1·74 on the C scale.

Ex. 2. Find the value of $2·3 \times 5·5$.

Placing the left-hand index of the C scale over 2·3 on the D scale we find that 5·5 on the C scale is off the rule. In a case like this we use the right-hand index of the C scale and place it over 2·3 on the D scale, then under 5·5 on the C scale the product 12·65 is read off on the D scale.

The beginner might imagine, by looking at the rule, that the last product should be 1·265; it is therefore *very important to find the position of the decimal point.* This is best done by approximating; in Ex. 2, the product is approximately $2 \times 5 = 10$ and therefore the answer must be 12·65 and not 1·265. Rules will however be given hereafter in Art. 252.

In working examples on continued multiplication, the Cursor is of great use.

Ex. 3. Find the value of $1{\cdot}2 \times 1{\cdot}8 \times 2{\cdot}3$.

Place the index of C over 1·2 on D, then move the cursor till it is over 1·8 on C, the product of these two numbers is then under the cursor. Without reading off this product, again move C until its index is under the cursor and then the final product 4·97 is read off on D under 2·3 on C.

Ex. 4. Find the value of $2{\cdot}1 \times 3{\cdot}9 \times 2{\cdot}4$.

Place the index of C over 2·1 on D, then move the cursor till it is over 3·9 on C; the partial product is on D under the cursor.

If now, as in the last example, we again move C until its left-hand index is under the cursor, we find that 2·4 on C is *off the rule.* All that we have to do in a case of this sort is to move C so that its right-hand index is under the cursor, then the final product 19·66 is read off on D under 2·4 on C.

[The product is approx. $2 \times 4 \times 2 = 16$ and therefore the decimal point is as given.]

244. *Division.*

Ex. Divide 2·1 by 1·7.

Place 1·7 on the C scale over 2·1 on the D scale, the quotient 1·235 is then read off on the D scale under the left-hand index of C.

If the left-hand index is off the rule, the right-hand index of C is used instead.

245. *Proportion.*

To find one term of a proportion given the other three.

Find x, if $$1{\cdot}72 : 8{\cdot}7 = x : 3{\cdot}49.$$

Here obviously $$x = \frac{1{\cdot}72 \times 3{\cdot}49}{8{\cdot}7}.$$

Divide 1·72 by 8·7 by placing 8·7 on the C scale over 1·72 on the D scale, then move the cursor so that it is over the right-hand index (the left-hand index being off the rule) of the C scale, *i.e.* over the quotient on the D scale.

We now have to multiply by 3·49 and do this by moving the C scale so that its left-hand index is under the cursor and the final value of x is read off on D under 3·49 on C.

It is found to be ·69.

$$\left[\text{The value of } x \text{ is approx. } \frac{2 \times 3}{9} = \frac{2}{3} = {\cdot}66.\right]$$

246. *Combined Multiplication and Division.*

Ex. Find the value of $\frac{2{\cdot}43\times1{\cdot}72\times7{\cdot}6}{3{\cdot}42\times2{\cdot}59\times8{\cdot}71}$.

Divide 2·43 by 3·42 by placing 3·42 on the C scale over 2·43 on the D scale; the quotient is on the D scale under the right-hand index of the C scale.

Multiply by 1·72 by **moving the cursor** to 1·72 on the C scale; the product is on the D scale under the cursor.

Divide by 2·59 by **moving the C scale** so that 2·59 on the C scale is under the cursor; the quotient is on the D scale under the right-hand index of the C scale.

Multiply by 7·6 by **moving the cursor** to 7·6 on the C scale; the product is on the D scale under the cursor.

Divide by 8·71 by **moving the C scale** so that 8·71 on the C scale is under the cursor; the final quotient is on the D scale under the right-hand index of the C scale.

The final result is ·412.

$$\left[\text{The approx. value is } \frac{2\times2\times8}{3\times3\times9}=\frac{32}{81}=\cdot4.\right]$$

Squares and Square Roots.

247. Since the distance from the index to any graduation on the C or D scale is double the distance from the index to the same graduation or the A or B scale, it follows that if any distance on the C or D scale represents $\log x$, the same distance on the A or B scale represents $2\log x$ or $\log x^2$.

Thus above any graduation on the D scale will be found its square on the A scale.

Ex. 1. Find the square of 2·27.

Place the cursor over 2·27 on the D scale; it will then be found to be over 5·15 on the A scale.

Thus $\qquad 2{\cdot}27^2=5{\cdot}15.$

[Approx. value is $2^2=4$.]

Ex. 2. Find the square of 178·5.

Place the cursor over 178·5 on the D scale, it will then be found to be over 31900 on the right-hand A scale.

$$\therefore\ 178{\cdot}5^2=31900=3{\cdot}19\times10^4.$$

[Approx. value is $180^2=32400$.]

248. In finding square roots, the following rules determine which scales to use.

1. If the number >1 mark off periods of two digits from the decimal point to the left, and ascertain how many digits are left in the last period marked:

If the number <1, ascertain how many significant figures there are in the first period to the right of the decimal point containing significant figures.

2. If this period contains *one* digit, use the *left-hand* A scale (since in this case the first figure of the square root cannot be greater than 3, and must therefore be found on the left-hand half of D).

3. If this period contains *two* digits, use the *right-hand* A scale.

Or, if the number be written as a multiple of a power of 10, then

i. The *left-hand* A scale is used if this power is *even*;

ii. The *right-hand* A scale is used if this power is *odd.*

e.g. $77{\cdot}5 = 7{\cdot}75 \times 10^1$. Use the right-hand A scale;
$\cdot000757 = 7{\cdot}57 \times 10^{-4}$. Use the left-hand A scale.

Ex. 3. Find the square root of 77·5.

Here there are an even number of digits in the last period marked and we therefore use the right-hand A scale.

Place the cursor over 77·5 on the A scale, and it is then over 8·8 on the D scale,

$$\therefore \sqrt{77{\cdot}5} = 8{\cdot}8.$$

[Approx. value $= \sqrt{81} = 9$.]

Ex. 4. Find the square root of ·000757.

Since there is one significant figure in the first period containing significant figures, the left-hand A scale is used.

Place the cursor over ·000757 on the A scale, and it will then be over ·0275 on the D scale,

$$\therefore \sqrt{\cdot000757} = \cdot0275 = 2{\cdot}75 \times 10^{-2}.$$

[Approx. value $= \sqrt{\cdot0009} = \cdot03$.]

Cubes and Cube Roots.

249. Having seen how to find the square of a number, we merely have to multiply this square by the number itself, and thus obtain the *cube*.

Ex. 1. Find the cube of 114·2.

Place the left-hand index of C opposite 114·2 on D; the number on the A scale opposite this index is obviously the square of 114·2.

Now multiply by 114·2 again, by looking at the number on the A scale opposite 114·2 on the B scale; we find $1490000 = 1{\cdot}49 \times 10^6$.

[Approx. value $= 110^3 = 1331 \times 10^3 = 1{\cdot}331 \times 10^6$.]

For alternative methods, see Art. 257, Exs. i, iii, v, ix.

250. By reversing this process we obtain *Cube Roots.* The slide must be moved until the number on the B scale under the given cube on the A scale is the same as the number on the D scale under the index on the C scale.

The following rules determine which of the scales on A and B are to be used:

1. If the number >1, mark off periods of 3 digits from the decimal point to the left and ascertain how many digits are left in the last period marked.

2. If the number <1, ascertain the number of significant digits in the first period of 3 digits, containing significant figures, to the right of the decimal point.

3. If this period contains *one digit,* use left-hand of A and left-hand of B.

4. If this period contains *two digits,* use right-hand of A and left-hand of B.

5. If this period contains *three digits,* use right-hand of A and right-hand of B.

Or again, if the number be written as a multiple of a power of 10, then

i. If this power is a multiple of 3, use the left-hand of A and left-hand of B;

ii. If this power is $1+$a multiple of 3, use the right-hand of A and left-hand of B;

iii. If this power is $2+$ a multiple of 3, use the right-hand of A and right-hand of B.

Ex. 2. Find the cube root of 33·5.

Marking the periods from the decimal point to the left, the last period contains *two digits.*

Therefore use the right-hand A scale and left-hand B scale.

The cursor is placed over 33·5 on the right-hand of A and the slide moved to the right until the number on the left-hand of B under the

cursor is found to be the same as the number on D under the left-hand index of C.

We thus obtain $\sqrt[3]{33\cdot5} = 3\cdot215$.

[Approx. value $= \sqrt[3]{27} = 3$.]

To find the logarithm of a number.

251. Move the slide until the index on C is over the given number on D, then turn the whole slide-rule over and read the number on the middle set of graduations (reading from right to left) opposite the black mark in the notch. This gives the mantissa.

For alternative method, see Art. 257, Ex. ix.

Ex. Find log 3.

Move the slide until the left-hand index of C is over 3 on D.

Invert the slide-rule and ·477 is found opposite the black mark.

252. *Rule for determining the position of the decimal point in a product.* The number of *digits* in a product is the same as the sum of the digits in the two factors, if the multiplication is performed with the slide projecting to the left; while it is one less if the slide projects to the right.

If there are more than two factors, the same rule can be successively applied. Thus the sum of the digits of all the factors is obtained and 1 subtracted each time a multiplication is performed with the rule to the right.

N.B. If a number >1, then by the *number of digits* we mean the number of figures to the left of the decimal point.

If a number <1 and starts with cyphers, by the number of digits we mean the number of cyphers coming immediately after the decimal point.

Ex. To find the product of $2\cdot4 \times 3\cdot7 \times \cdot0059$.

Place the left-hand index of C over 2·4 on D and move the cursor to 3·7 on C. [The slide projects to the right.] Now multiply by ·0059 by placing the right-hand index of C under the cursor and read off the final product on D under ·0059 on C. [The slide projects to the left.]

The final product gives the figures 524 and we have to determine where to put the decimal point.

The number of *digits* in the original factors is $1+1-2=0$; from this we have to subtract 1, since *one* multiplication was performed with the slide projecting to the right.

Therefore number of *digits* in product is -1, and consequently the product is ·0524.

253. *Rule for determining the position of the decimal point in a quotient.*

The number of *digits* in a quotient is the same as the excess of the number of digits in the dividend over the number in the divisor, if the slide is projecting to the left; while it is one more if the slide projects to the right.

Ex. Divide 5·01 by ·0322.

Place ·0322 on C over 5·01 on D. [The slide projects to the right.] The quotient 1556 is then found under the left-hand index of C. To determine the position of the decimal point, we find, by the above rule, the number of digits in the quotient to be $1-(-1)+1=3$.

Therefore quotient $=155·6$.

EXAMPLES L.

Multiply

1. 7·42 by 8·5.
2. 3·45 by 71·2.
3. ·0431 by ·00728.
4. 825·7 by ·0241.

Evaluate

5. $43·17 \times 22·05 \times ·715$.
6. $·00295 \times 72·4 \times 15·8$.
7. $1324 \times 18·9 \times ·075$.

Divide

8. ·1125 by ·015.
9. 693 by 8·4.
10. 162·9 by ·517.
11. ·396 by ·0072.

Find the value of x in the following equations:

12. $72·1 : 8·2 = 3 : x$.
13. $52·7 : x = ·021 : 42·5$.
14. $x : 51·9 = 72·41 : 8·05$.
15. $17·2 : 15·1 = x : 92·7$.

Find the value of

16. $\dfrac{18·2 \times 19·5}{7·3 \times 14·21}$.
17. $\dfrac{137·5 \times 49·2 \times 81}{77·2 \times 59·6}$.
18. $\dfrac{52·41 \times 71·42 \times 1·41}{11·27 \times 15·3 \times 18·9}$.
19. $\dfrac{19·24 \times 17·81 \times 15·21}{202 \times 15·91 \times 18·24}$.

20. Find the squares of (i) 19·5. (ii) ·0527. (iii) 185·4. (iv) 324·9. (v) ·005843.

21. Find the square roots of (i) 85·5. (ii) 103·5. (iii) ·0724. (iv) ·0000895. (v) 8250.

22. Find the cubes of (i) 75·9. (ii) 821·5. (iii) ·035. (iv) ·0059.

23. Find the cube roots of (i) 72·8. (ii) 824·5. (iii) 7·98. (iv) ·00582. (v) ·000785.

Sines and Tangents.

254. *To find the sine of an angle.* (i) Invert the whole slide-rule and move the scale of sines until the necessary number of degrees comes opposite the black mark; then turning the whole slide-rule over again, the required value of the sine is found on B opposite the right-hand index of A.

If the result is found on the right-hand B scale, a decimal point is put at the beginning; while if it is found on the left-hand B scale, a cipher is first placed at the beginning and then the decimal point in front of the cipher.

Ex. To find sin 30°.

Turn the slide-rule over and draw out the slide until 30 on the Sine scale is opposite the black mark. Then we find 5 opposite the right-hand index of A.

Thus $\sin 30° = ·5$.

(ii) A second method is to take the slide right out and then put it back again with the Sine scale next to the A scale and its extremities coinciding with the extremities of the A scale. The sines of all the angles are then read off on the A scale opposite the corresponding number of degrees on the Sine scale.

Between 70° and 90° the graduations, if marked, would be extremely close together, so that only 75° and 80° are indicated. The sines of other angles between 70° and 90° may be obtained from any of the approximate rules found in books on the Slide Rule.

255. *To find the tangent of an angle.* (i) Invert the whole slide-rule and move the scale of tangents until the necessary number of degrees comes opposite the black mark; on turning the slide rule over again the value of the tangent is found on A opposite the right-hand index of B.

As in the case of the sines, a decimal point is prefixed, if the result is found on the right-hand A scale; a decimal point and a cipher if the result is on the left-hand A scale.

Ex. Find tan 5°.

Turn the slide-rule over and draw out the slide until 5 on the Tangent scale is opposite the black mark; we then find 875 opposite the right-hand index of the B scale.

Therefore tan 5° = ·0875.

(ii) The second method is to take the slide out and re-insert it with the Tangent scale next to the A scale and the extremities coinciding.

The tangents of all the angles up to 45° are then read off on the A scale opposite the corresponding angle on the Tangent scale.

For angles between 45° and 90°, we obtain the tangents from the formula

$$\tan A = \frac{1}{\cot A} = \frac{1}{\tan(90^\circ - A)},$$

$$e.g.\ \tan 60^\circ = \frac{1}{\tan 30^\circ}.$$

APPLICATIONS.

256. Ex. 1. Find the number of degrees in 2·57 radians.

$$1 \text{ radian} = 57{\cdot}3^\circ,$$

$$\therefore\ 2{\cdot}57 \text{ radians} = 57{\cdot}3^\circ \times 2{\cdot}57 = 147{\cdot}3^\circ.$$

[Place the right-hand index of C opposite 57·3 on D, then under 2·57 on C we read 147·3 on D.]

Ex. 2. Find the circumference of a circle when the diameter is 12 inches.

$$\text{Circumference} = \pi d = \pi \times 12$$
$$= 37{\cdot}7 \text{ inches.}$$

[Place the left-hand index of B opposite π (specially marked) on A; then opposite 12 on B we read 377 on A.]

Ex. 3. Find the area of a circle when the diameter is 6 inches.

$$\text{Area} = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 6^2 \text{ sq. inches} = 28{\cdot}3 \text{ sq. inches.}$$

[Divide π by 4 by placing 4 on B underneath π on A (the quotient is on A over the right-hand index of B); then multiply by d^2 by observing the reading on A opposite 6 on C. We obtain 283.]

Ex. 4. Find the volume of a sphere 5·7 cms. in radius.

$$\text{Volume} = \tfrac{4}{3}\pi r^3 = 4{\cdot}189 \times 5{\cdot}7^3 \text{ cu. cms. (since } \pi = 3{\cdot}142)^*$$
$$= 776 \text{ cu. cms.}$$

[Multiply 4·189 by 5·7 by moving the slide till the left-hand index of B is under 4·189 on A, then move the cursor to 5·7 on B. Multiply this result by $5{\cdot}7^2$ by moving the slide till the right-hand index is under the cursor, and the final result is on A opposite 5·7 on C.]

Ex. 5. Find the area of a triangle, the sides being 27·5, 22·4 and 19·8 cms. respectively.

$$a = 27{\cdot}5$$
$$b = 22{\cdot}4$$
$$c = 19{\cdot}8$$
$$2\underline{|69{\cdot}7.}$$
$$\therefore\ s = 34{\cdot}85,$$
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{34{\cdot}85 \times 7{\cdot}35 \times 12{\cdot}45 \times 15{\cdot}05} = 220 \text{ sq. cms.}$$

[Since we eventually have to take a square root, it will be convenient to work with the A and B scales.

Place the left-hand index of B on 34·85 of the left-hand A scale and the cursor on 7·35 of the left-hand B scale.

Move the left-hand index of B to the cursor, and then the cursor to 12·45 on the left-hand B scale.

Move the left-hand index of B to the cursor and the final product of the four factors is found on A opposite 15·05 on B. By a rough calculation the product contains 5 factors and is therefore 48000.

To find the square root, we place the cursor over 48000 on the left-hand A scale (since there is an odd number of digits) and find it is then over 220 on D.]

Ex. 6. Find B and C given that $b = 17{\cdot}2$, $c = 15{\cdot}4$ and the included angle $A = 38° 40'$.

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$
$$= \frac{1{\cdot}8}{32{\cdot}6}\cot 19° 20' = \frac{1{\cdot}8}{32{\cdot}6} \times \frac{1}{\tan 19° 20'} = {\cdot}157.$$

* In finding the volumes of spheres, it will in future be advisable to remember that

$$\tfrac{4}{3}\pi = 4{\cdot}189.$$

[Inverting the slide-rule and placing 19° 20′ on the Tangent scale opposite the black mark, then turning the slide-rule over, we read ·351 opposite the right-hand index of the B scale,

$$\therefore \tan\frac{B-C}{2}=\frac{1{\cdot}8}{32{\cdot}6}\times\frac{1}{{\cdot}351}.$$

Place 32·6 on the C scale opposite 1·8 on the D scale; the quotient is then on the D scale, under the right-hand index on the C scale.

Put the cursor at this place and then move the slide until ·351 on the C scale is under the cursor; the final result ·157 is then found on the D scale under the left-hand index of C.]

To obtain $\frac{B-C}{2}$, we move the slide until the right-hand index of B is under ·157 on the right-hand A scale; turning the rule over and looking at the black mark against the Tangent scale, we find that

$$\frac{B-C}{2}=8^\circ 55'.$$

Now $$\frac{B+C}{2}=90^\circ-\frac{A}{2}=70^\circ 40',$$

$$\therefore\ B=79^\circ 35',$$

$$C=61^\circ 45'.$$

Ex. 7. Given that 1 inch = 2·54 centimetres, find the number of centimetres in 537 inches.

Place the right-hand index of C opposite 2·54 on D, then under 537 on C we read 1364 on D;

$\therefore$ 537 inches = 1364 centimetres.

Ex. 8. Find $3\frac{1}{2}$ % of $115\frac{1}{2}$.

$$3\tfrac{1}{2}\,\% \text{ of } 115\tfrac{1}{2}=\frac{115{\cdot}5\times 3{\cdot}5}{100}=1{\cdot}155\times 3{\cdot}5=4{\cdot}04.$$

[Place the left-hand index of C opposite 1·155 on D, then under 3·5 on C we read 4·04 on D.]

Ex. 9. Find the space fallen through (in vacuo) by a body in 27 seconds.

$$s=\tfrac{1}{2}gt^2=16\times 27^2 \text{ ft.}=11660 \text{ ft. (approx.)}.$$

[Place the left-hand index of C opposite 27 on D, then opposite 16 on B we find 11660 on A.]

257. Mr A. G. Thornton, S. Mary's Street, Manchester, is now selling a new Slide Rule called the "Improved Perry Calculating Slide Rule." It has very many advantages; we

will here consider some of the special advantages attaching to the Log Log Scales which are marked on it.

Between the edge and the scale A is another scale called E, in which the markings are proportional to the logarithm taken twice of each number.

Thus the position of 10 is the zero position, for $\log\log 10 = 0$, and 10 is placed at a convenient point of the scale, then 4 is placed to the *left* of 10 at a distance proportional to $\log\log 4$ or $-\cdot 2204$; 50 is placed to the *right* of 10 at a distance proportional to $\log\log 50$ or $\cdot 2303$ and so on.

Between the other edge and the scale D is another scale called E^{-1} on which the graduations are the reciprocals of those on E, thus 4 on E corresponds with ·25 on E^{-1}, 50 on E with ·02 on E^{-1} and so on.

The following are the most important types of calculation, and the student who has the Rule in his hands will readily follow the method of working.

(i) Calculate x from $x = 2{\cdot}31^{1\cdot 32}$.

Set B, 1 on E, 2·31 then find B, 1·32 and read off the answer E, 3·02.

Reason.
$$\log x = 1{\cdot}32 \log 2{\cdot}31,$$
$$\therefore \log\log x = \log 1{\cdot}32 + \log\log 2{\cdot}31.$$

(ii) Calculate x from $x = 2{\cdot}31^{-1\cdot 32}$.

Proceed just as in (i) but opposite B, 1·32 read off the answer E^{-1}, ·33.

Reason. We really calculate as in (i) and read off the reciprocal of the answer.

(iii) Calculate x from $x = \cdot 568^{1\cdot 52}$.

Set B, 1 on E^{-1}, ·568 then find B, 1·52 and read off the answer E^{-1}, ·423.

Reason. It is not possible to take $\log\log \cdot 568$; we have therefore to use the reciprocal $\frac{1}{\cdot 568}$, the process is then the same as in (i) except that the reciprocal scale E^{-1} takes the place of E all through, thus

$$\log\log\frac{1}{x} = \log 1{\cdot}52 + \log\log\frac{1}{\cdot 568}.$$

(iv) Calculate x from $x = \cdot 568^{-1\cdot 52}$.

Set B, 1 on E^{-1}, ·568 then find B, 1·52 and read off the answer

E, 2·36.

Reason. We really calculate as in (iii) and read off the reciprocal of the answer.

(v) Calculate x from $x = 2\cdot 31^{\frac{1}{1\cdot 32}}$.

Set B, 1·32 on E, 2·31 then find B, 1 and read off the answer

E, 1·89.

Reason.
$$\log x = \frac{1}{1\cdot 32} \log 2\cdot 31,$$
$$\therefore \log\log x = \log\log 2\cdot 31 - \log 1\cdot 32.$$

(vi) Calculate x from $x = 2\cdot 31^{-\frac{1}{1\cdot 32}}$.

Proceed just as in (v) but opposite B, 1 read off the answer

E^{-1}, 53.

Reason. We really calculate as in (v) and read off the reciprocal of the answer.

(vii) Calculate x from $x = \cdot 568^{\frac{1}{1\cdot 32}}$.

Set B, 1·32 on E^{-1}, ·568 then find B, 1 and read off the answer

E^{-1}, ·652.

Reason. It is not possible to take $\log\log \cdot 568$; we have therefore as in (iii) to use reciprocals. Thus we proceed exactly as in (v) using the E^{-1} instead of the E scale throughout.

(viii) Calculate x from $x = \cdot 568^{-\frac{1}{1\cdot 32}}$.

Proceed just as in (vii) but opposite B, 1 read off the answer

E, 1·54.

Reason. We really calculate as in (vii) and read off the reciprocal of the answer.

(ix) Calculate x from $x = \log_{1\cdot 32} 2\cdot 31$.

i.e. Solve $1\cdot 32^x = 2\cdot 31$.

Set B, 1 on E, 1·32 then find E, 2·31 and read off the answer

B, 3·01.

Reason. $\log x = \log\log 2\cdot 31 - \log\log 1\cdot 32.$

EXAMPLES LI.

1. Find the number of degrees in 7·2 radians.

2. Calculate the number of radians in 62°.

3. Find the number of sq. centimetres in a circle of radius 5·8 cms.

4. What is the number of centimetres in the circumference of a circle of radius 7·2 cms.?

5. Find the volume of a sphere of radius 13·2 decimetres.

6. Calculate the number of degrees in 3·4 radians.

7. Obtain the circumference of a circle of diameter 7 centimetres.

8. Find the area of a circle of diameter 16 centimetres.

9. Find the number of radians in 140°.

10. If the volume of a sphere is 18560 cu. centimetres, what is the radius?

11. Find the volume of a sphere when the radius is 15·9 centimetres.

12. Calculate the radius of a circle whose area is 1000 sq. centimetres.

13. Find the area of a triangle when the sides are 15, 17·5 and 19·5 centimetres respectively.

14. Calculate the angles B and C of a triangle, given that $b=7·5$, $c=3·2$ and $A=50°$.

15. If there are 1·609 kilometres in 1 mile, how many kilometres are there in 827 miles?

16. Calculate the value of 23 tons, if 1 lb. = 2·205 kilograms.

17. Find the number of centimetres in 5 miles, if 1 ft. = 30·48 cms.

18. Find the values of the angles C and A of a triangle, if $c=18·75$, $a=14·21$ and $B=74°50'$.

19. Calculate the area of a triangle, the sides of which are 24·7, 59·8 and 62·5 centimetres respectively.

20. Given that the earth's radius is $6·371\times 10^8$ centimetres, find its value in miles, when 1 foot = 30·48 cms.

21. Find the mass of the earth in tons, given that it is $6·14\times 10^{27}$ grams, and that 1 lb. = 453·6 grams.

TABLES OF LOGARITHMS, SINES, ETC.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes				
											1′	2′	3′	4′	5′
0°	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
45°	7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	4	4
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	9998	9999	9999	9999	9999	9999	9999	9999	9999	9999	0	0	0	0	0

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	14
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	6	10	13	17
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	10	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	10	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
45°	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	23	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	$\underline{9}210$	$\underline{9}292$	$\underline{9}375$	$\underline{9}458$	$\underline{9}542$	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	$\bar{0}057$	$\bar{0}145$	$\bar{0}233$	$\bar{0}323$	$\bar{0}413$	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	74	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	118
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2·9042	9208	9375	9544	9714	9887	$\bar{0}061$	$\bar{0}237$	$\bar{0}415$	$\bar{0}595$	29	58	87	115	144
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	162	203
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	214	267
77	4·3315	3662	4015	4374	4737	5107	5483	$\underline{5}864$	$\underline{6}252$	$\underline{6}646$	62	124	186	248	310
78	4·7046	7453	7867	8288	8716	9152	9594	$\bar{0}045$	$\bar{0}504$	$\bar{0}970$	73	146	219	292	365
79	5·1446	1929	2422	2924	3435	3955	$\underline{4}486$	$\underline{5}026$	$\underline{5}578$	$\underline{6}140$	87	175	262	350	437
80	5·6713	7297	7894	8502	9124	9758	$\bar{0}405$	$\bar{1}066$	$\bar{1}742$	$\bar{2}432$					
81	6·3138	3859	4596	5350	6122	6912	7920	8548	9395	$\bar{0}264$					
82	7·1154	2066	3002	3962	4947	5958	6996	$\underline{8}062$	$\underline{9}158$	$\bar{0}285$					
83	8·1443	2636	3863	5126	6427	7769	9152	$\bar{0}579$	$\bar{2}052$	$\bar{3}572$					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

The value of the tangent here increases so rapidly that *common* difference columns cannot be used.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
0°	Inf. Neg.	7·2419	5429	7190	8439	9408	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$961					
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	21	41	62	82	103
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	8·9403	9489	9573	9655	9736	9816	9894	9970	0046	0120	13	26	39	52	65
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
45°	9·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	9·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	9·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	9·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	9·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	9·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	9·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	9·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	9·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	9·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	9·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	9·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	9·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	9·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	9·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	9·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	9·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	9·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	9·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	9·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	9·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	9·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	9·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	9·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	9·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	9·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	9·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	9·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	9·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	9·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	9·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	9·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	9·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	9·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	9·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	9·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	9·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	9·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	9·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	9·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	9·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	9·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	9·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	9·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	9·9999	9999	0000	0000	0000	0000	0000	0000	0000	0000	0	0	0	0	0
90	10·0000														

											Minutes				
		6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	Inf. Neg.	7·2419	5429	7190	8439	9409	$\bar{0}200$	$\bar{0}870$	$\bar{1}450$	$\bar{1}962$					
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	29	58	87	116	145
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	21	41	62	83	103
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	8·9420	9506	9591	9674	9756	9836	9915	9992	0068	$\bar{0}143$	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
45°	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	10·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	10·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	10·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	10·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	10·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	10·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	10·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	10·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	10·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	10·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	10·9784	9857	9932	0008	0085	0164	0244	0326	0409	0494	13	26	40	53	66
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	20	41	62	83	103
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	29	58	87	116	144
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	11·7581	8038	8550	9130	9800										
	12					0591	1561	2810	4571	7581					

ANSWERS.

PART II.

EXAMPLES XXXV. (page 244).

30. 100·65 feet. **31.** $2:1,\ 1:3,\ 3:2$.

32. 4·773 centimetres.

EXAMPLES XXXVII. (page 262).

1. 19·59 sq. centimetres.

2. 9·6 cms.; 7·4 cms.; 5·0 cms.

EXAMPLES XXXVIII. (page 266).

1. 8·6605 cms.; 259·82 sq. cms.

2. 13·254 ft. **3.** 5 cms.

4. 105·804 sq. inches. **7.** 181 sq. cms.

8. 173 sq. ins. **9.** 9·5 cms.

12. 114588 sq. ft.

EXAMPLES XXXIX. (page 273).

N.B. In some of these answers more than sufficient is given, *e.g.* in 11, $\frac{2n\pi}{5}$ is sufficient as it embraces $2n\pi$.

1. $\frac{n\pi}{2}+(-1)^n\frac{\pi}{6}$. **2.** $\frac{2n\pi}{3}\pm\frac{\pi}{9}$.

3. $\frac{n\pi}{4} + \frac{\pi}{16}$.

4. $36n^\circ + (-1)^n 4^\circ 6'$.

5. $60n^\circ \pm 3^\circ 2'$.

6. $\frac{180n^\circ}{7} + 5^\circ 1'$.

7, 8, 9. $\frac{n\pi}{3} \pm \frac{\pi}{9}$.

10. $2n\pi$; $\frac{2n+1}{3}\pi$.

11. $(2n\pi)$; $\frac{2n\pi}{5}$.

12. $n\pi$.

13. $\frac{n\pi}{6}$; $\frac{n\pi}{4}$.

14. $(n\pi)$; $\frac{n\pi}{7}$.

15. $\left(\frac{n\pi}{2}\right)$; $\frac{n\pi}{4}$.

16. $2n\pi - \frac{\pi}{2}$; $\frac{2n\pi}{5} + \frac{\pi}{10}$.

17. $\frac{n\pi}{4} + \frac{\pi}{16}$; $n\pi + \frac{\pi}{4}$.

18. $\frac{n\pi}{9} + \frac{\pi}{18}$.

19. $\frac{n\pi}{3}$; $2n\pi \pm \frac{\pi}{3}$.

20. $\frac{n\pi}{4}$; $\frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$.

21. $\frac{n\pi}{2}$; $n\pi + (-1)^n 14^\circ 29'$.

22. $\frac{n\pi}{2}$; $n\pi \pm 22^\circ 21'$.

23. $\left(2n\pi \pm \frac{\pi}{2}\right)$; $\frac{n\pi}{4}$; $\left(\frac{2n+1}{4}\pi\right)$.

24. $2n\pi \pm \frac{\pi}{2}$; $\frac{n\pi}{6} + \frac{\pi}{24}$; $\frac{n\pi}{2} - \frac{\pi}{8}$.

25. $\frac{n\pi}{8}$; $\left(\frac{n\pi}{4}\right)$.

26. $\left(\frac{n\pi}{4}\right)$; $\frac{n\pi}{12}$.

27. $\frac{n\pi}{2}$; $\frac{(2n+1)\pi}{8}$.

28. $n\pi$; $\frac{2n+1}{10}\pi$.

29. $360n^\circ + 63^\circ 50'$; $360n^\circ - 20^\circ 14'$.

30. $360n^\circ + 74^\circ 44'$; $360n^\circ - 33^\circ 38'$.

31. $360n^\circ + 36^\circ 52'$.

32. $360n^\circ + 31^\circ 54'$; $2n\pi$.

33. $2n\pi + \frac{\pi}{3}$; $(2n-1)\pi$.

34. $2n\pi + \frac{\pi}{4}$.

35. $2n\pi + \frac{5\pi}{12}$; $2n\pi - \frac{\pi}{12}$.

36. $2n\pi + \frac{\pi}{2}$; $(2n+1)\pi + \frac{\pi}{6}$.

37. $n\pi + \frac{\pi}{4}$; $2n\pi$; $2n\pi + \frac{\pi}{2}$.

38. $\frac{n\pi}{2}$; $\frac{n\pi}{2} \pm \frac{\pi}{12}$.

39. $2n\pi \pm \dfrac{\pi}{2}$; $n\pi \pm 41^\circ\, 24'$. 40. $n\pi \pm \dfrac{\pi}{3}$.

41. $(4n+1)\dfrac{\pi}{2}$; $(4n \pm 1)\,\pi$. 42. $2n\pi \pm \dfrac{\pi}{3}$.

43. $\dfrac{n\pi}{4} + (-1)^n\, 7^\circ\, 30'$. 44. $n\pi + \dfrac{\pi}{12}$; $n\pi + \dfrac{5\pi}{12}$.

45. $\dfrac{(2n+1)\,\pi}{4} + \dfrac{3\pi}{16}$. 46. $n\pi \pm \dfrac{\pi}{6}$; $n\pi \pm \dfrac{\pi}{4}$; $n\pi$.

47. $2n\pi$; $n\pi + \dfrac{\pi}{4}$. 48. $n\pi$; $\dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{4}$.

49. $\dfrac{n\pi - \alpha - \beta}{3}$. 50. $2n\pi \pm \dfrac{5\pi}{6}$.

51. $2n\pi \pm \dfrac{2\pi}{3}$. 52. $2n\pi$; $2n\pi \pm \dfrac{\pi}{2}$; $2n\pi \pm \dfrac{\pi}{3}$.

53. $n\pi + (-1)^n \dfrac{\pi}{6}$; $4n\pi + \dfrac{\pi}{2}$; $4n\pi - \dfrac{3\pi}{2}$.

54. $n\pi + \dfrac{\alpha}{2}$; $\dfrac{(2n+1)\,\pi}{6} - \dfrac{\alpha}{6}$. 55. $n\pi - \dfrac{\alpha + \beta}{2}$.

Examples XL. (page 288).

7. $\sin\dfrac{A}{2} = \dfrac{3}{5}$, $\cos\dfrac{A}{2} = -\dfrac{4}{5}$.

8. $\sin\dfrac{A}{2} = -\dfrac{8}{17}$, $\cos\dfrac{A}{2} = -\dfrac{15}{17}$.

9. $\sin 130^\circ = 0{\cdot}7660$, $\cos 130^\circ = -\,0{\cdot}6428$.

10. $\sin 115^\circ = 0{\cdot}9063$, $\cos 115^\circ = 0{\cdot}4226$.

11. $\frac{3}{4}$, $-\frac{4}{3}$.

Examples XLI. (page 296).

1. $\frac{12}{5}$. 2. $\frac{3}{4}$. 3. $\frac{63}{65}$. 4. $\frac{56}{65}$.

EXAMPLES XLII. (page 298).

1. $\frac{1}{6}$ or -1.

2. a or a^2-a+1.

3. 2.

4. $\dfrac{-1\pm\sqrt{b^2+2}}{a}$.

5. $\sqrt{\dfrac{10\pm4\sqrt{2}}{17}}$.

6. $\dfrac{ab\{\sqrt{a^2-1}-\sqrt{b^2-1}\}}{a^2-b^2}$.

7. 2 or -1.

8. 13.

9. $\dfrac{\sqrt{3}}{2}$.

10. 0, $\frac{1}{2}$.

11. $\dfrac{3\sqrt{3}\pm\sqrt{7}}{8}$.

12. $\frac{156}{5}$.

13. $\frac{1}{4}$.

14. $\sqrt{3}$, $-(2+\sqrt{3})$.

15. ab.

16. $x=\dfrac{a+b+c\pm\sqrt{a^2+b^2+c^2-ab-bc-ca+3}}{3}$.

EXAMPLES XLIII. (page 303).

1. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

2. $x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=1$.

3. $b^2(h^2+k^2)=(a^2-ch)^2$.

4. $a^2+b^2=1$.

5. $y(x^2-1)=2$.

6. $(x^2-y^2)^2+16xy=0$.

7. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

8. $xy^2=4a^3$.

9. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

10. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}$.

11. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

12. $\dfrac{x^2}{a}+\dfrac{y^2}{b}=a+b$.

13. $\dfrac{b^2}{a^2}-\dfrac{bd}{ac}=2$.

14. $\cot\alpha=\dfrac{1}{a}-\dfrac{1}{b}$.

15. $ab(ab-4)=(a+b)^2\tan^2\alpha$.

16. $8bc=a\{4b^2+(b^2-c^2)^2\}$.

17. $\tan^2\alpha=\tan^2\beta+\tan^2\gamma$.

18. $8(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$

19. $(a^2+b^2)(c-1)+2b(c+1)=0.$

20. $\dfrac{a(m+b)}{\sqrt{(n+b)^2+(m+b)^2}}=\dfrac{mn-b^2}{n+b}.$

21. $(x-y)^{\frac{2}{3}}+(x+y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$

22. $m^{\frac{2}{3}}+n^{\frac{2}{3}}=(mn)^{-\frac{2}{3}}.$

23. $(x\cos a+y\sin a)^{\frac{2}{3}}+(x\sin a-y\cos a)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}.$

24. $a^2+c^2-2ac\cos 2\phi=b^2.$

25. $a(x^2+y^2)+2a^3+6a^2x=\pm(3a^2+2ax)^{\frac{3}{2}}.$

Examples XLIV. (page 313).

1. ·0014544, ·9999990.
2. $\frac{1}{13}$ radian.
3. ·0008727, ·9999996.
4. ·0040891 radian.
5. $\frac{1}{29}$ radian.
6. $\frac{1}{17}$ radian.
7. ·011547 radian.
8. $\frac{1}{37}$ radian.
9. – ·000513 radian.
10. $\frac{1}{21}$ radian.
11. 3′·72.
12. ·86353.
13. 13·06 metres.
14. 15·8 metres.

Examples XLV. (page 320).

1. $\dfrac{\sin\dfrac{3n+1}{2}\mathrm{A}\sin\dfrac{3n\mathrm{A}}{2}}{\sin\dfrac{3\mathrm{A}}{2}}.$

2. $\dfrac{\cos n\mathrm{A}\sin n\mathrm{A}}{\sin\mathrm{A}}.$

3. $\dfrac{\cos\dfrac{3n-1}{6}\mathrm{A}\sin\dfrac{n\mathrm{A}}{2}}{\sin\dfrac{\mathrm{A}}{2}}.$

4. $\dfrac{\cos\left\{\theta+\dfrac{(n-1)\pi}{2n}\right\}}{\sin\dfrac{\pi}{2n}}.$

5. 0.

6. $\dfrac{\sin\dfrac{n+1}{4}\mathrm{A}\sin\dfrac{n}{4}\mathrm{A}}{\sin\dfrac{\mathrm{A}}{4}}.$

7. $\dfrac{\sin^2 \dfrac{10\pi}{21}}{\sin \dfrac{\pi}{21}}$.

8. $\dfrac{\cos \dfrac{11\pi}{23} \sin \dfrac{11\pi}{23}}{\sin \dfrac{\pi}{23}}$.

9. $\dfrac{\sin^2 \dfrac{n\pi}{2n-1}}{\sin \dfrac{\pi}{2n-1}}$.

10. $\dfrac{\sin \left(\dfrac{n+1}{2} a + \dfrac{n-1}{2} \pi\right) \sin \dfrac{n(a+\pi)}{2}}{\sin \dfrac{a+\pi}{2}}$.

11. $\dfrac{\cos \left\{(n+1) a + \dfrac{n-1}{2} \pi\right\} \sin \dfrac{n(2a+\pi)}{2}}{\sin \dfrac{2a+\pi}{2}}$.

12. $\dfrac{\sin \left(2a + \dfrac{n^2-1}{2n} \pi\right) \sin \dfrac{n+1}{2} \pi}{\sin \dfrac{n+1}{2n} \pi}$.

13. $\dfrac{\cos \left\{3a + \dfrac{(n-1)^2}{2n} \pi\right\} \sin \dfrac{n-1}{2} \pi}{\sin \dfrac{n-1}{2n} \pi}$.

14. $\dfrac{n}{2} \cos 2a - \dfrac{\cos (n+1) 2a \sin 2na}{2 \sin 2a}$.

15. $\dfrac{n}{2} \cos 2a + \dfrac{\cos (n+1) 2a \sin 2na}{2 \sin 2a}$.

16. $\dfrac{\sin (2n+3) \theta \sin 2n\theta}{\sin 2\theta} - \dfrac{n \sin 3\theta}{2}$.

17. $\dfrac{\tan (2n+1) a - \tan a}{\sin 2a}$.

18. $\dfrac{\cot a - \cot (3n+1) a}{\sin 3a}$.

19. $\dfrac{\tan 2 (n+1) a - \tan 2a}{\sin 2a}$.

20. $\dfrac{\cot 2\alpha - \cot (n+2)\alpha}{\sin \alpha}$.

21. $\dfrac{n}{2} - \dfrac{\cos (2\alpha + \overline{n-1}\,\beta) \sin n\beta}{2 \sin \beta}$.

22. $\dfrac{n}{2} + \dfrac{\cos (n+3)\alpha \sin n\alpha}{\sin \alpha}$.

23. $\dfrac{n}{2}$.

24. $\dfrac{3 \cos \left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{4 \sin \frac{\beta}{2}} + \dfrac{\cos 3 \left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{3n\beta}{2}}{4 \sin \frac{3\beta}{2}}$.

25. $\dfrac{3 \sin \frac{n+1}{2}\alpha \sin \frac{n\alpha}{2}}{4 \sin \frac{\alpha}{2}} - \dfrac{\sin \frac{3(n+1)}{2}\alpha \sin \frac{3n\alpha}{2}}{4 \sin \frac{3\alpha}{2}}$.

26. $\frac{1}{8}[3n - 4 \cos (n+1)\alpha \sin n\alpha \operatorname{cosec} \alpha$
$+ \cos (2n+2)\alpha \sin 2n\alpha \operatorname{cosec} 2\alpha]$.

27. $\dfrac{1}{8}\Big[3n + 4 \cos 2n\alpha \sin \dfrac{3n\alpha}{2} \operatorname{cosec} \dfrac{3\alpha}{2}$
$+ \cos 4n\alpha \sin 4n\alpha \operatorname{cosec} 4\alpha\Big]$.

28. $-\sin \dfrac{n\alpha}{n-2}$.

29. $\frac{1}{2} \operatorname{cosec} \theta \{\tan (n+1)\theta - \tan \theta\}$.

30. $\dfrac{\sin \left(\frac{n+1}{2}\alpha + \frac{n-1}{2}\pi\right) \sin \frac{n(\pi+\alpha)}{2}}{\sin \frac{\pi+\alpha}{2}}$.

31. $\dfrac{1}{4} \sin \dfrac{n\alpha}{2}\left[\cos \dfrac{n-1}{2}\alpha + \cos \dfrac{n+3}{2}\alpha + \cos \dfrac{n+7}{2}\alpha\right] \operatorname{cosec} \dfrac{\alpha}{2}$
$+ \dfrac{1}{4} \sin \dfrac{3n\alpha}{2} \cos \dfrac{3n+9}{2}\alpha \operatorname{cosec} \dfrac{3\alpha}{2}$.

32. $\tan^{-1}(n+1)x - \tan^{-1} x$.

33. $\tan^{-1}(n+1) - \dfrac{\pi}{4}$.

EXAMPLES XLVII. (page 340).

1. $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

2. $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

3. $2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$.

4. $5\{\cos(\tan^{-1}\frac{3}{4})+i\sin(\tan^{-1}\frac{3}{4})\}$

or $$5\{\cos 36^\circ\, 52'+i\sin 36^\circ\, 52'\}.$$

5. $\sqrt{298}\,\{\cos(\tan^{-1}\frac{17}{3})+i\sin(\tan^{-1}\frac{17}{3})\}$

or $$\sqrt{298}\,\{\cos 80^\circ+i\sin 80^\circ\}.$$

6. $\sqrt[4]{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$; $\sqrt[4]{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$;

$$\sqrt[4]{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right);\quad \sqrt[4]{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right).$$

7. $\sqrt{29}\,(\cos 21^\circ\, 48'+i\sin 21^\circ\, 48')$;
$\sqrt{29}\,(\cos 141^\circ\, 48'+i\sin 141^\circ\, 48')$;
$\sqrt{29}\,(\cos 261^\circ\, 48'+i\sin 261^\circ\, 48')$.

8. $k\left(\cos\frac{5\pi}{72}+i\sin\frac{5\pi}{72}\right)$; $k\left(\cos\frac{7\pi}{72}+i\sin\frac{7\pi}{72}\right)$;
$k\left(\cos\frac{9\pi}{72}+i\sin\frac{9\pi}{72}\right)$; $k\left(\cos\frac{11\pi}{72}+i\sin\frac{11\pi}{72}\right)$;
$k\left(\cos\frac{13\pi}{72}+i\sin\frac{13\pi}{72}\right)$; $k\left(\cos\frac{15\pi}{72}+i\sin\frac{15\pi}{72}\right)$;

where $$k=(\sqrt{6}-\sqrt{2})^{\frac{1}{6}}.$$

9. $\cos(8\theta-9\phi)+i\sin(8\theta-9\phi)$.

10. $\cos(9\theta+7\phi)-i\sin(9\theta+7\phi)$.

11. $\cos 16\theta+i\sin 16\theta$.

12. $\cos 20\theta+i\sin 20\theta$.

EXAMPLES XLVIII. (page 347).

1. ·8414710
   ·5403023.

2. $\cos a - h\sin a - \frac{h^2}{\underline{\lvert 2}}\cos a + \frac{h^3}{\underline{\lvert 3}}\sin a + \ldots$

3. $(-1)^n \frac{3^{2n}+3}{4\underline{\lvert 2n}}\theta^{2n}$.

4. $\theta = \frac{1}{21}$ radian.

5. $(-1)^n \frac{2^{2n}(1-2^{2n-1})}{\underline{\lvert 2n+1}}\theta^{2n+1}$.

6. $\cos a$.

7. $-\frac{16}{15}$.

8. $-\frac{1}{30}$.

9. $\frac{5}{3}$.

10. $-\frac{a^2+ab+b^2}{ab}$.

11. 2.

18. $1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4 + \frac{61}{720}\theta^6 + \frac{277}{8064}\theta^8$.

TEST PAPERS.

XLVI. (page 350).

6. $\tan A \tan B$ or q.

XLVII. (page 351).

3. 2297 yards.

4. 2·903 metres.

5. 7575 sq. metres.

XLVIII. (page 352).

5. $B = 99° 35'$; $C = 55° 24'$; $b = 4997$,
or $B = 30° 23'$; $C = 124° 36'$; $b = 2564$.

6. 6087 ft.

XLIX. (page 353).

1. B $= 37° 18'$; C $= 83° 1'$; $c = 11060$.
7. $120°$.

LII. (page 355).

1. 1·152 miles.
2. $\dfrac{4n-1}{2}\pi$; $\dfrac{4n+1}{10}\pi$.
6. $n\pi$; $n\pi \pm 26° 34'$.

LIII. (page 356).

1. $(2n+1)\pi$, $n\pi \pm \dfrac{\pi}{6}$.
3. 1730 metres.

LIV. (page 357).

2. (i) $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$.

   (ii) $\dfrac{n\pi}{4} + (-1)^n \dfrac{\pi}{12}$.
4. 50·1 metres.
5. $15° 38'$.
6. 39740 sq. cms.

LV. (page 358).

1. $x = \frac{3}{2}$.
3. $n\pi + 8° 51'$.

LVI. (page 359).

2. (i) $\pm \cdot 1602$; $\pm 6\cdot 2432$.

   (ii) $\dfrac{2n+1 \pm \sqrt{4n^2+4n-15}}{4}$.
4. $\dfrac{-1 \pm \sqrt{2+b^2}}{a}$.

MISCELLANEOUS EXAMPLES (page 361).

7. $\theta=(2n+1)\frac{\pi}{2}$, $\frac{4n+1}{14}\pi$, $\frac{4n-1}{6}\pi$.

24. $\theta=(2n+1)\frac{\pi}{3}$, $\frac{4n+3}{2}\pi$.

36. $\frac{12}{35}$.

40. $n\pi+\frac{\pi}{4}$, $2n\pi\pm\frac{\pi}{3}$.

52. $(2n+1)\frac{\pi}{10}$, $(2n+1)\frac{\pi}{2}$.

56. $2n\pi+2\tan^{-1}\frac{b\pm\sqrt{a^2+b^2-c^2}}{a+c}$.

59. $45°$.

65. 458·2575 feet, 103923 sq. ft.

67. (i) $2n\pi$, $\frac{2}{3}\left(n\pi+\tan^{-1}\frac{b}{a}\right)$; (ii) $\frac{4n+1}{10}\pi$, $\frac{4n-1}{2}\pi$.

71. 285·4 feet.

78. $2n\pi-\alpha$, $\frac{4n-1}{2}\pi+\alpha$.

85. $(2n+1)\frac{\pi}{6}$, $\frac{2n\pi+\alpha+\gamma\pm\beta}{3}$.

102. $(a^2+b^2)(a^2+b^2-3)-2b=0$.

121. $x=41370$.

126. $x=-\cdot835$

EXAMPLES XLIX. (page 383).

1. $9{\cdot}493210\times10^6$.
2. $3{\cdot}198466\times10^{-5}$.
3. $3{\cdot}426098\times10^2$.
4. $2{\cdot}013538\times10^{-5}$.
5. $4{\cdot}062639\times10^{-2}$.
6. $4{\cdot}107103\times10$.
7. $1{\cdot}987772$.
8. $6{\cdot}021347\times10^{-1}$.
9. $5{\cdot}120802\times10^2$.
10. $6{\cdot}629801\times10^{-3}$.
11. $3{\cdot}914192\times10^{-1}$.
12. $8{\cdot}17765\times10^{-4}$.
13. $4{\cdot}07824\times10$.
14. $4{\cdot}700017\times10^{-3}$.
15. $1{\cdot}56827\times10^{-4}$.

EXAMPLES L. (page 393).

1. 63·1. 2. 245·6. 3. ·000314.
4. 19·9. 5. 681. 6. 3·38.
7. 1877. 8. 7·5. 9. 82·5.
10. 315. 11. 55. 12. ·341.
13. 107000. 14. 467. 15. 105·5.
16. 3·42. 17. 119. 18. 1·619. 19. ·0889.
20. (i) $3{\cdot}8 \times 10^2$. (ii) $2{\cdot}78 \times 10^{-3}$. (iii) $3{\cdot}438 \times 10^4$. (iv) $1{\cdot}06 \times 10^5$. (v) $3{\cdot}4 \times 10^{-5}$.
21. (i) 9·25. (ii) $1{\cdot}02 \times 10$. (iii) $2{\cdot}69 \times 10^{-1}$. (iv) $9{\cdot}46 \times 10^{-3}$. (v) $9{\cdot}08 \times 10$.
22. (i) $4{\cdot}36 \times 10^5$. (ii) $5{\cdot}54 \times 10^8$. (iii) $4{\cdot}288 \times 10^{-5}$. (iv) $2{\cdot}05 \times 10^{-7}$.
23. (i) 4·18. (ii) 9·38. (iii) 1·998. (iv) $1{\cdot}8 \times 10^{-1}$. (v) $9{\cdot}225 \times 10^{-2}$.

EXAMPLES LI. (page 400).

1. 412·4°. 2. 1·082 radians.
3. 106 sq. cms. 4. 45·2 cms.
5. 9640 cu. dms. 6. 194·7°.
7. 22 cms. 8. 201 sq. cms.
9. 2·44 radians. 10. 16·43 cms.
11. 16850 cu. cms. 12. 17·84 cms.
13. 125·7 sq. cms. 14. $B = 105° 45'$, $C = 24° 15'$.
15. 1331 kiloms. 16. 113600 kilogs.
17. 805000 cms. 18. $C = 62° 48'$, $A = 42° 22'$.
19. 735·2 sq. cms. 20. 3960 miles.
21. $6{\cdot}04 \times 10^{21}$ tons.

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